

THE INDIAN MENSURATION

BY

RAM NATH CHATTERJEE, M. A.,

LECTURER, PROFESSOR OF MATHEMATICS, MUIR CENTRAL COLLEGE ;

AND

DEPUTY MAGISTRATE, BENGAL

Allahabad :
INDIAN PRESS
1890.

(The right of Translation and reproduction is reserved.)

P R E F A C E.

MENSURATION, in a scheme of University education, should not be taught by merely coaching up the students in a number of dry, unconnected, and arbitrary rules. They cannot excite any interest, and do not evoke in the student an intelligent appreciation of the subject. The University of Allahabad, therefore, has made a departure in the right direction in providing in the Regulations for the Entrance Examination that "the course in Mensuration includes so much as presupposes a knowledge of the first four books of Euclid." A text-book meeting the requirements of the above Regulation is felt to be a decided want both by teachers and students. The books at present used by them invariably begin with the properties of Similar Triangles and the ratios of Geometrical Magnitudes, which assume on the part of the student a knowledge of the Sixth Book of Euclid.

To supply this want is the main object of this little book. A cursory glance will show that the characteristic feature of this book is that no rule has been given in it which has not at the same time been proved by the aid of the first four books of Euclid only.

Another important object which the book is intended to serve is to provide the student with a sufficient number of practical exercises on the propositions of Euclid, so that the book may be regarded as a sequel to Euclid.

The book is divided into two parts. The first part deals with matter which assumes a knowledge of the first two books of Euclid only, and may be used by the student immediately after he has passed the Middle Class Anglo-Vernacular Examination. The second part, which treats of Circles and Regular Polygons and requires a knowledge of the third and fourth books of Euclid, may be studied later on.

The exercises, which are numerous and varied, have been selected with care to suit the capacities of all classes of students from the most elementary to the far advanced.

I have to tender my sincere thanks to Dr. Thibaut for much kind assistance in the revision of the first portion of the proof-sheets. I am also heavily indebted to my pupils Raghbir Prasad and Shyam Lal of the Muir Central College for the help I have received from them in the tedious work of verifying the answers to the Exercises.

MUIR CENTRAL COLLEGE,
ALLAHABAD, 11th August, 1890.

} RAM NATH CHATTERJEE

CONTENTS.

INTRODUCTION	1
--------------	-----	-----	-----	-----	---

PART I.

The Rectangle	7
EXERCISES I	9
PRACTICAL METHODS	10
EXERCISES II	12
Do. III	13
Do. IV	17
Right-angled Triangles	20
EXERCISES V	22
PRACTICAL METHODS	23
EXERCISES VI	28
Triangles (<i>generally</i>)	33
EXERCISES VII	37
Do. VIII	41
Parallelograms	45
EXERCISES IX	46
Trapeziums	48
EXERCISES X	50
Quadrilaterals (<i>generally</i>)	52
EXERCISES XI	53
Irregular Rectilineal Figures	55
EXERCISES XII	56
On angles	59
EXERCISES XIII	60

Miscellaneous Propositions	62
EXERCISES XIV	68
<i>Miscellaneous Examples</i>	70

PART II.

Tangents and Chords of a circle	85
EXERCISES XV	92
In-circle, Ex-circle and Circum-circle	94
EXERCISES XVI	97
Inscribed and Circumscribed Regular Polygons.			98
EXERCISES XVII	106
Circumference of a circle	108
EXERCISES XVIII	110
MISCELLANEOUS PROPOSITIONS	113
EXERCISES XIX	120
EXAMPLES WORKED OUT	123
<i>Miscellaneous Examples</i>	126

CORRIGENDA.

Page.

- 12 Line 8 for 11 sq. in., read 41 sq. in.
- 13 „ 14 „ 72" „ 12"
- 19 Exer. 19 for border of yds., read a border of five yards.
- 28 for Exercises V read Exercises VI.
- 29 „ 4 for 88 & 110 ft. read 88 & 110 yds.
- 38 Line 21 for $AM^2 + 2CM.MD$ read $AM^2 - 2CM.MD$.
- 60 for Exercises XII read Exercises XIII.
- 67 Line 18 for $x(a+x)$ read $x(a+h)$.
- 68 for Exercises XIII read Exercises XIV.
- 75 Exer. 48. for one-fourths read three-fourths.
- 81 „ 108 for 4 *ac* read 2 *ac*.
- 92 for Exercises XIV read Exercises XV.
- 93 Last line, after division inset at right-angles to the diameter.
- 95 Line 10 for $r=s-c$ read $r=s-a$.
- 98 11 for $\sqrt{\frac{3}{2}}$ read $\sqrt{\frac{3}{2}}$.
- 105 for $\frac{2I}{\sqrt{4-R^2}}$ read $\frac{2I}{\sqrt{4-I^2}}$
- 106 „ 15 for $I' =$ read $I =$
- 112 „ 4 for diameter and area read diameter.
- 147 . Life last but one Ans. Exer. 8 for 121 miles read 111 miles.
- 154 Line 4 Ans. Exer. 15 for 999.24 in., read 99.924 yds.

THE INDIAN MENSURATION.

INTRODUCTION.

1. The science of **Mathematics** treats of *quantities*, or things that can be measured.

2. **Mensuration** is that department of Mathematics which is occupied with the investigation of rules by which the lengths, areas, volumes and other dimensions of bodies are calculated from the measurement of simple associated magnitudes.

Mensuration (from Lat. *mensura*, measure) is a branch of applied Geometry. It gives rules for finding the lengths of lines, the areas of surfaces, or the volumes of solids, from certain simple data of lines and angles. We may call it, therefore, the science of *indirect measurement*. Thus if I want to find the height of the Vizianagram Tower in the Muir Central College, I may go to the top, and let a string with a weight attached to its end down to the bottom, and then measure the length of the string; but this is not *mensuration*, though it is *measurement*. But if I measure the length of the shadow of the tower, say at nine in the morning, and measure the shadow of my walking stick at the same time, and apply the geometrical theorem that the height of the tower bears the same ratio to the length of my stick as the shadow of the tower bears to that of my stick, and thus find the height of the tower indirectly, this is an application of Mensuration. Again in many cases direct measurements are impossible, and measurements must be effected indirectly. How, for example, is the height of Mount Everest above the sea-level to be found? How is the area of the vast continent of India to be ascertained? Such results must be ascertained by the measurement of certain simple associated magnitudes, *e. g.*, the length of a base line and the magnitudes of certain angles.

A fairly complete treatise on Mensuration, giving all the methods of indirect measurement would require on the part of the student an acquaintance

with Trigonometry and the Infinitesimal Calculus. But as this elementary book is intended principally for junior students, we shall presuppose only a knowledge of the first four books of Euclid and the simpler portions of Elementary Algebra.

3. Units of Length and Area. The Numerical Measure of any concrete quantity (such as a line, an area, an angle, &c.) is its ratio to a selected standard quantity of the same kind as itself, called the **Unit**. Thus in measuring lengths we take for our unit a yard, a cubit, a foot, or a metre.

The British standard unit of length is the Imperial Yard. This is defined to be the distance between the centres of the lines engraved on gold plugs, sunk in a bar of bronze, kept at the Exchequer Chambers, and known as the Imperial standard yard, the temperature of the bar at the time of observation being 62° Fahrenheit.

A general standard of linear or square measurement did not exist in British India until recently. "In some of the local offices the standard measure is simply a matter of tradition and when applied for, the Nazir of the Court is directed to report on the correct length of the *kath* or *luggee*—this he does with the utmost simplicity by holding up his own arm, pointing from the elbow to the tip of the little finger, sometimes adding that as he is a small made man, one, two or four, fingers' breadth must be added on." *Thuillier*. It was not uncommon in Behar to find a landlord and a tenant disputing over the selection of the individual whose arm is to be taken as the standard *kath*. It is extremely desirable, therefore, that all measurements should be made by the *acre*. A step has already been taken in this direction.

By Act II. of 1889 (The Measures of Length Act), it has been enacted that the imperial standard yard for the United Kingdom shall be the legal standard measure of length in British India and be called the standard. One-third part of the standard yard shall be called a standard foot, and one thirty-sixth part of such a yard shall be called a standard inch.

The **Unit of Area** is derived from the Unit of Length. Thus we commonly define the **Unit of Area** to be the area of the

square described upon the unit of length. For example, if a foot be taken as the unit of length, the square whose side is one foot is the unit of area.

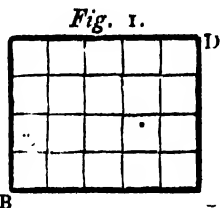
The unit angle used in this book is a *degree*; which is the 90th part of a right angle.

3. Symbolical expression for concrete quantities.
It will be evident from what has been said, that the *complete* mathematical representation of a concrete quantity always consists of two factors, one being the Unit of the same kind as the quantity considered, and the other representing the number of times such units are contained in the quantity. Hence, if one foot be the unit of length the length of a line which contains a units is a times one foot or a feet. The line is simply denoted by the symbol a , the factor—one *foot*—being understood. Similarly the area of a field which contains A units of area, is $A \times$ one square foot, *i. e.*, A square feet. And the area is denoted simply by the symbol A .

4. Area of a Rectangle.

Let $ABCD$ be a rectangle; and let the side AB contain a units of length and the side DC , b units of length.

Divide AB into a equal parts each equal to the unit of length, and divide BC into b equal parts each equal to the unit of length. Through these points of division draw straight lines parallel to the sides of the rectangle. These lines divide the rectangle into a number of squares, each of which is the unit of area (see art 2). As we have a rows of these squares, and b squares in each row, the total number of these squares, or units of area, in the rectangle $ABCD$ will be a taken b times, or ab .



Obs. In Fig. 1 AB is divided into four equal parts and BC into five equal parts. The area, therefore, is twenty units of area, or simply 20.

It is in the sense indicated above that we say that the area of a rectangle is equal to the product of its two adjacent sides, represented respectively by numbers denoting the number of units of length contained in each side. It ought to be remembered that the unit in the numerical products is no longer the linear unit, but the square unit, or the unit of area.

The student ought early to guard himself against the notion that it is possible arithmetically to multiply a concrete number by another concrete number. Abstract numbers can be multiplied together, as 4 times 5 is 20; a concrete number can be multiplied by an abstract number, as 5 times 4 feet is 20 feet; but we cannot multiply a concrete number or an abstract number by a concrete number; for example, we cannot say 4 feet times 5, or 4 feet times 5 feet. Again, with regard to division, a concrete number can be divided by another concrete number of the same kind, as 20 feet contains 4 feet 5 times; but we cannot divide an abstract number by a concrete number.

In saying that the area of a rectangle is equal to the product of its two adjacent sides, we no longer speak of arithmetical multiplication, but simply extend the meaning of the term multiplication. Thus, one concrete quantity is spoken of as multiplied by another concrete quantity, when the arithmetical multiplication of the measures of these concrete quantities gives us the measure of some other concrete quantity. This last concrete quantity so formed is always derived or defined by reference to the former quantities. Thus a cubic foot is defined by reference to a square foot and a linear foot.

TABLES.

LINEAR MEASURE.

12 inches (12 in.)	make	1 foot (1 ft.) •
3 feet	1 yard (1 yd.)	
5½ yards	1 rod, or pole (1 po.)	
40 poles or 220 yards	1 furlong (1 fur.)	
8 furlongs	}	1 mile (1 m.)
or		
1760 yards		
or		
5280 feet		

Also

A cubit	}	= 18 inches
		or 1½ feet.
A fathom	.	= 6 feet.
A cable's length	.	= 120 fathoms.
A league	.	= 3 miles.
A knot (nautical)	}	= 6080 feet,
or		
A geographical mile	.	= 60 knots
A degree of latitude	.	or 69½ miles.
A link	.	= 7.92 inches
A chain	.	= 100 links
	.	= 792 inches
	.	= 66 feet
	.	= 22 yards.

SQUARE MEASURE,

144 sq. inches	make	1 sq. foot (1 sq. ft.)
9 sq. feet		1 sq. yard (1 sq. yd.)
30$\frac{1}{4}$ sq. yards		1 sq. rod, sq. pole, or perch (1 per.)
40 perches		1 rood (1 ro.)
4 roods		1 acre (1 ac.)
640 acres		1 sq. mile (1 sq. m.)

Also

1 acre	= 10 sq. chains
	= 4840 sq. yards
	= 10,0000 sq. links

PART FIRST.

[***In this Part we confine ourselves to those rules of Mensuration which presuppose a knowledge of the first two Books of Euclid.]

SECTION I.

THE RECTANGLE.

6. Given the sides of a Rectangle, to find its area.

It has already been proved (art. 4), that if the measures of the lengths of the two adjacent sides of a rectangle $ABCD$ be a and b respectively, the measure of the area is equal to the product of a and b .

Therefore area $ABCD = ab$ units of area $= ab$.

RULE 1. The area of a rectangle = product of the lengths of its sides.

Example 1.—Let $a = 49$ ft. 6 inches, and $b = 25$ feet, the area of the rectangle $= 49.5 \times 25 = \underline{1237.5}$ square feet.

Example 2.—Find the number of square feet in the floor of a rectangular room 25 ft. 3 in. long and 13 ft. 6 in. broad.

$$25.25 \times 13.5 = \underline{340.875} \text{ sq. ft.}$$

Example 3.—Find the rent of a field whose length is 50 ch. 62 lks., and breadth 23 ch. 14 lks., at Rs. 25 per acre.

$$50.62 \times 23.14 = 1171.3468 \text{ sq. chains} = 117.13468 \text{ acres.}$$

$$\text{The rent} = \text{Rs. } 25 \times 117.13468 = 292.8367 = \text{Rs. } \underline{117-10 \text{ as. } 9 \text{ p.}}$$

nearly.

RULE 2. The area of a rectangle divided by its length, gives the breadth; and the area divided by the breadth gives the length.

Example 1.—The area of a rectangular field is 20 acres 5 chains and $42\frac{1}{2}$ links, and one side is 825 links; find the other side.

$$20 \text{ ac. } 5 \text{ ch. } 42\frac{1}{2} \text{ links} = 20.5425 \text{ acres} = 2054250 \text{ sq. links.}$$

$$\text{Again } \frac{2054250}{825} = 2490 \text{ links} = \underline{24 \text{ chains } 90 \text{ links.}}$$

7. The area of a Square. A square being a rectangle whose sides are equal we can at once determine its area if the length of its side be given. Thus, if the length of its side = a , the area = $a \times a = a^2$. ●

RULE 3. The area of a square = the square of the side.

Example 1.—How many square yards are there in the floor of a room 15 feet square?

$$\text{Area} = 15 \times 15 \text{ square feet} = 225 \text{ square feet} = \underline{25 \text{ square yards.}}$$

Example 2.—The length of a field is equal to twice its breadth, and the rent of the field at Rs. 2-8 an acre is Rs. 24-8 as.; find the dimensions of the field.

$$\text{The area} = \frac{24\frac{8}{12}}{2\frac{1}{2}} \text{ acres} = \frac{49}{5} \text{ acres} = 9.8 \text{ acres.}$$

Let a be the breadth in yards, then the length = $2a$ yards.

$$\text{And } a \times 2a \text{ square yards} = 9.8 \text{ acres.}$$

$$= 9.8 \times 4840 \text{ sq. yards.}$$

$$\therefore 2a^2 = 9.8 \times 4840$$

$$\text{or } a^2 = 49 \times 484$$

$$\text{or } a = 7 \times 22$$

$$\therefore \text{The breadth is } \underline{154 \text{ yards.}}$$

$$\text{And the length is } \underline{308 \text{ yards.}}$$

EXERCISES I.

1. Find the area of a rectangle 65 yards long, and 25 yards broad.

2. How many square feet and inches are there in a rectangular table which is 12 feet 6 inches long, and 5 feet 3 inches broad?

3. The side of a square is 36 inches; required its area in square feet.

4. The base of the Great Pyramid in Egypt is a square, the side of which is 764 feet; find the number of acres, roods and perches it occupies.

5. The side of a square is 10 chs. 48 lks., find the area.

6. The length of an oblong being 34 chs. 56 lks., and the breadth 22 chs. 64 lks., what is the area?

7. Find in acres and decimals of an acre the area of a rectangular *maidan*, 32 chs. 45 lks., by 45 chs. 28 lks.

8. A square garden contains 10 acres; find the length of its side.

9. The area of a rectangular field is 19 acres, and its length is 1496 yards; what is the breadth?

10. The sides of three squares are 112, 144, and 320 yards respectively; required the length of the side of a square that shall be equal in area to the three given squares.

11. How many square inches are there in the floor of a room 17 feet square?

12. What is the difference between 5 ft. square and 5 square ft.?

13. What is the length of a floor which is 24 feet broad, and equal in area to a floor 36 feet square?

14. A table which is $12\frac{1}{2}$ inches wide contains $6\frac{1}{4}$ square feet; what is its length?

15. What is the area of a square table, the side of which is 5 feet 6 inches?

16. If a chess-board measures one foot square, what is the area of each of its divisions?

17. Find the difference between the area of a square, the side of which measures 16 yards, and that of a rectangle 35 feet long, and 33 feet broad.

18. What length of plank $5\frac{1}{2}$ inches wide will make a square foot?

19. The area of a rectangular field is 7 ac. 3 ro. 10 po., and one side is double the other; find the sides.

20. A sheet of "Double Super Royal" paper measures 40 inches by $27\frac{1}{2}$; how many square yards would a ream of it cover?

21. A grass plot, which is 216 feet long and 196 feet wide contains 10 square flower beds, the sides of which are each 9 feet; find the number of square yards of grass.

22. What length must be cut off from a rectangular piece of land 385 yards wide that the area of the part cut off may be 7 acres?

23. A field can be divided into one million squares, each measuring a cubit along a side; find the area of the field.

24. Find the area in acres swept over by an Engine in travelling from Howrah to Dehli a distance of 954 miles, the gauge or distance between the rails being 5 feet 6 inches.

25. The Senate-Hall in the Muir Central College is 82 feet long by 36 feet broad. How many candidates can be examined in this Hall allowing 20 square feet for each desk and stool?

PRACTICAL METHODS.

8. When the dimensions of a rectangular field are given in links, the area comes out in square links. If we divide

the number by 100,000, or, which is the same thing, if from the number we cut off the last five figures, the remaining figure towards the left hand gives the number of acres in the area, the remainder being decimal parts of an acre. This latter being multiplied by 4, the integral part gives the roods, and the decimal part being multiplied by 40 gives the perches.

Example 1.—Suppose the sides of rectangle to be 347 chains and 42 chains 50 links; what is the area ?

$$\begin{aligned}
 347 \text{ chs.} &= 34700 \text{ links.} \\
 42 \text{ chs. } 50 \text{ lks.} &= 4250 \text{ links.} \\
 34700 \times 4250 &= 147475000 \text{ square links.} \\
 &= 1474.75 \text{ acres.} \\
 &= \underline{1474 \text{ acs. } 3 \text{ roods.}}
 \end{aligned}$$

Example 2.—Suppose the side of a rectangle to be 8 chs. 48 lks. and 3 chs. 65 lks; what is the area?

$$848' \times 365 = 309520 \text{ square links.}$$

cutting off 5 figures from the right hand

gives Acr. 3.0952

$$\text{Rd. } \frac{4}{0.3808}$$

$$\text{Per. } \frac{40}{15.2320}$$

Hence area = 3 Ac. 0 rd. 15 per.

9. **Duodecimals or cross-multiplication**, is a rule made use of by builders, glaziers and other workmen in computing the contents of their works. The following table, in which the linear as well as the superficial foot is divided and subdivided duodecimally (*i.e.*, into twelfths) will explain the process of cross-multiplication.

1 foot (linear or superficial) = 12 primes.

1 prime (1') = 12 seconds.

1 second (1'') = 12 thirds.

1 third (1''') = 12 fourths.

and so on—

Example 1.—Express, 11 ft. $5\frac{1}{4}$ in. as a duodecimal.

$$11 \text{ ft. } 5\frac{1}{4} \text{ in.} = \underline{11 \text{ ft. } 5' 3''}.$$

Example 2.—Express 45 sq. ft. 41 sq. inches in duodecimals.

$$\begin{aligned} 45 \text{ sq. ft. } 11 \text{ sq. in.} &= 45 \frac{41}{144} \text{ sq. ft.} \\ &= \left(45 + \frac{36}{144} + \frac{5}{144}\right) \text{ sq. ft.} \\ &= \left(45 + \frac{3}{12} + \frac{5}{144}\right) \text{ sq. ft.} \\ &= \underline{45 \text{ sq. ft. } 3' 5''} \end{aligned}$$

Example 3.—Express 525 sq. ft. 5' 7" 6''' in sq. yds. &c.

$$\begin{aligned} 525 \text{ sq. ft. } 5' 7'' 6''' &= \left(525 + \frac{5}{12} + \frac{7}{144} + \frac{6}{12} \text{ of } \frac{1}{144}\right) \text{ sq. ft.} \\ &= \underline{58 \text{ sq. yds. } 3 \text{ sq. ft. } 67\frac{1}{2} \text{ sq. in.}} \end{aligned}$$

EXERCISES II.

Express as duodecimals.

1. 4 ft. $7\frac{1}{2}$ in. 2. 7 ft. $2\frac{3}{4}$ in. 3. 5 sq. ft. $131\frac{1}{2}$ sq. in.
4. 9 sq. ft. $36\frac{1}{2}$ sq. in. 5. 12 sq. yds. 3 sq. ft.
6. 93 sq. ft. $132\frac{1}{2}$ sq. in. 7. 3 sq. yds. 5 sq. ft. $113\frac{1}{2}$ sq. in.

carpet. The length, breadth and width must, of course, be reduced to the same denomination—yards, feet or inches—before we employ the processes of multiplication and division.

RULE—Length of Carpet: $\frac{\text{length of room} \times \text{breadth of room}}{\text{width of carpet}}$

Example.—What length of carpet 30 inches wide will cover a floor 12 ft. 6 in. by 7 ft. 6 in.?

Length of carpet 37½ feet.

§—**Papering the Walls of a Room.** To find the area of the walls of a rectangular room, we imagine the four walls opened out, in other words, we imagine them as standing side by side; we then shall see that the four walls make up a rectangle, the length of rectangle being = twice (length of room + breadth of room) and breadth = height of room. The following diagram explains the above reasoning.

height 1st wall 2nd wall 3rd wall 4th wall height

length breadth length breadth

RULE 1.—Area of the walls = $2 \times (\text{length} + \text{breadth}) \times \text{height}$.

= perimeter of floor \times
height of room

Def.—Perimeter = sum of the length of the bounding lines.

NOTE.—In finding the area of the walls for the purpose of papering, deductions must be made for doors, windows, fire-lace &c.

RULE 2.—Length of paper = $\frac{\text{area of the walls}}{\text{width of paper}}$

Example 1.—Find the area of the walls of a room 16 ft. 6 in. long, 12 ft. 6 in. broad, and 14 ft. high.

$$\text{Perimeter} = 2 \times (\text{length} + \text{breadth})$$

$$= 2 (16\frac{1}{2} + 12\frac{1}{2})$$

$$= 58 \text{ ft.}$$

$$\text{height} = 14 \text{ ft.}$$

$$\text{area of wall} = 58 \times 14 \text{ ft.}$$

$$= \underline{812 \text{ sq. ft.}}$$

Example 2.—If in the room in *Ex. 1*, there are two doors 7 ft. by 4 ft., and 2 windows 6 ft. by 3 ft., how many yards of paper 2 ft. 6 in. wide, will be required?

$$\text{area of doors} = 2 \times 7 \times 4 = 56 \text{ sq. ft.}$$

$$\text{area of windows} = 2 \times 6 \times 3 = 36 \text{ sq. ft.}$$

$$\text{area of paper required} = 812 - (56 + 36) = 720 \text{ sq. ft.}$$

$$\text{length of paper} = \frac{720}{2\frac{1}{2}} = 288 \text{ feet} = \underline{32 \text{ yards.}}$$

12.—Paving an area with stone slates.—The area of stones must be the same as that of the surface to be paved. If then the stones be all of uniform dimensions,

$$\text{Number of slates} = \frac{\text{area to be paved}}{\text{area of each stone}}$$

Ex. How many stone slates, each 3 ft. long, 2 ft. 6 in. wide, will pave a courtyard 75 yards long, 40 yds. 1 ft. broad?

$$\text{Number of slates} = \frac{225 \times 121}{2 \times 2\frac{1}{2}} = \underline{3630}$$

13.—Area of uniform path running round (immediately inside) the boundary of rectangular court.

Let ABCD be a rectangular court a feet long and b feet broad, and let a uniform path c feet wide run round it. The path encloses a smaller rectangular PQRS whose length is $(a - 2c)$ feet, and breadth $(b - 2c)$ feet.

$$\begin{aligned}
 \text{Hence the area of the path} &= \text{area ABCD} - \text{area PQRS} \\
 &= ab - (a - 2c)(b - 2c) \\
 &= c(2a + 2b) - 4c^2 \\
 \text{or} &= \underline{2c(a + b - 2c)}.
 \end{aligned}$$

Note.—Here the path takes off from the area of the given court.

Area of uniform path running round (immediately outside) the boundary of garden.

In fig. 2. Let PQRS be the garden and ABCD the outer boundary of the path. Let $PQ = u$, $QR = v$, and width of path $= w$.

$$\begin{aligned}
 \text{Then area of the path} &= (u + 2w)(v + 2w) - uv \\
 &= \underline{2w(u + v + 2w)}.
 \end{aligned}$$

Example 1.—A rectangular park is 256 yards long, 164 yards broad; how much ground will be occupied by a road 30 feet wide running round (immediately inside) the boundary of the park?

$$\text{Area of park} = (256 \times 164) \text{ sq. yds.}$$

The road takes off $(10 + 10)$ yds. from the length and $(10 + 10)$ yds. from the breadth.

$$\begin{aligned}\therefore \text{Area of road} &= 256 \times 164 - 236 \times 144 \\ &= \underline{8000 \text{ sq. yds.}}\end{aligned}$$

or directly by the formula—

$$\begin{aligned}\text{Area} &= 2c(a+b-2c) \\ &= 2 \times 10(256+164-20) \\ &= \underline{8000 \text{ sq yds.}}\end{aligned}$$

Example 2.—A castle occupies a rectangular plot of ground 743 yards long and 531 yards broad, and is surrounded by a moat 13 yards wide; find the superficial area of the moat

$$\begin{aligned}\text{Area of moat} &= (743+26)(531+26) - 743 \times 531 \\ &= \underline{33800 \text{ sq. yards.}}\end{aligned}$$

or directly by the formula—

$$\begin{aligned}\text{Area} &= 2w(u+v+2w) \\ &= 2 \times 13 \times (743+531+26) \\ &= \underline{33800 \text{ sq. yds}}\end{aligned}$$

EXERCISES IV.

1. The main passenger platform of the Allahabad Railway Station is 1274 ft. long by 21 ft. 8 in. broad. Find the cost of paving it with Burdwan paving stones at Rs. 25 per 100 sq. feet. What will be the number of such stones each measuring $2\frac{1}{2}' \times 1\frac{1}{4}'$?
Find also the number of Magh-mela passengers that can be admitted to the platform, allowing on an average a standing-ground of $2\frac{1}{2}$ square feet to each passenger.
2. How many yards of carpeting, $\frac{3}{4}$ yard wide, will be required to cover the floor of the Muir Central College Library measuring 55 feet by 34 feet?

3. How many yards of paper 15 inches wide will be required to cover the walls of a room 45 yards 1 foot 3 inches in circuit, and 10 feet 8 inches in height?

4. A rectangular cricket-field 250 yards in length and 210 yards in breadth is surrounded by a gravel walk 5 feet wide : find the area of the walk.

5. A square field contains $62\frac{1}{2}$ acres : what is the length of its side?

6. The area of a rectangular bowling green is an acre, and its length and breadth are in the ratio of 5 : 2 ; around this green is a path whose width is 2 yards : find how many bricks will pave the path, allowing 50 bricks for a square yard?

7. A room is 30 feet long, 12 feet 6 inches broad, and 10 feet high. It has two doors each 8 feet high and 3 feet 4 inches wide, and two windows each 8 feet 4 inches high and 5 feet wide. How many pieces of paper, each 10 yards long and 1 yard wide, will be required to paper its walls?

8. A rectangular garden, 150 yards long and 64 yards wide, is surrounded by a wall 7 feet 6 inches high ; find the cost of white-washing the walls on the inner side at 6 annas per 100 square feet.

9. Find the cost of plastering the walls and ceiling of a room 25 feet long, 18 feet wide and 11 feet high : the walls being charged at the rate of 3 annas per square yard, and the ceiling at 7 annas per square yard.

10. In a cricket-field 150 yards long and $144\frac{1}{2}$ yards broad a square plot equal to $\frac{1}{12}$ of the area of the field is kept for cricket ; find the side of this square, and the expense of keeping the field at 1 anna per square yard.

11. If 42 pieces of paper, each containing 6 yards, are used in papering the walls of a room 25 feet 2 inches long and 19 feet

10 inches wide, and 10 feet 6 inches high, what is the width of the paper?

12. A rectangular park, the sides of which are in the ratio of 1 : 4, contains 640 acres : how much ground will be occupied by a road, 16 feet wide, running round immediately inside the boundary of the park?

13. How many square feet of deal-board will be required to make a rectangular packing case 6 feet 8 inches long, 4 feet 6 inches broad, and 4 feet deep?

14. Find the difference in area of a floor 30 feet long and 24 feet broad, and two others of half these dimensions.

15. The expense of carpeting a room whose breadth is 8 feet 4 inches with carpet 18 inches wide at Rs. 4-8 as. a yard is Rs. 225 ; find the length of the room.

16. The length of a room is 25 feet ; the cost of painting the four walls at Rs. 2-8 as. per square yard is Rs. 400 and that of carpeting the room at Rs. 5 per square foot is Rs. 2,500 : find the height and breadth of the room.

17. A rectangular field which is half as long again as it is broad contains 15 acres : how long will it take a man to walk 4 times round it at the rate of 3 miles an hour?

18. India has an area of 1,474,910 square miles ; how many acres does it contain, and how many Bengal Bighas? (1 Bengal bigha = 1,600 square yards).

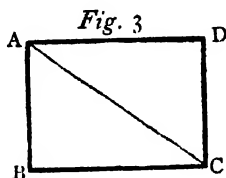
19. The sides of a rectangular plot of ground are 393 feet and 150 feet : shew that a border of yards must be taken off all round that the area left may be an acre.

20. The areas of the floor and walls of a room are respectively 375 square feet, 300 square feet, and 180 square feet. Find the dimensions of the room.

SECTION II.

RIGHT-ANGLED TRIANGLES.

14. The diagonal AC of the rectangle ABCD divides the rectangle into two right-angled triangles ABC and ACD which are equal in area (Euc. I. 34). Therefore the area of the right-angled triangle ABC



$$= \frac{1}{2} \times \text{the area of the rectangle ABCD}$$

$$= \frac{1}{2} \times AB \times BC.$$

RULE. Area of a right-angled triangle = half the product of the sides containing the right angle.

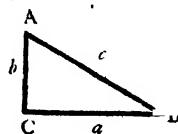
Example. The sides about the right-angle of a right-angled triangle are 50 yards and 34 yards respectively: find the area.

$$\text{Area} = \frac{1}{2} \times 50 \times 34 = \underline{850} \text{ square yards.}$$

15. The Pythagorean Relation:—

Fig. 4.

Let ABC be a right-angled triangle, having the angle ACB right. The square on AB is equal to the sum of the squares on BC and CA. (Euc. I. 47).



Let a, b, c be the measures of the sides opposite to the angles A, B, C respectively,

$$\text{then } c^2 = a^2 + b^2 \dots\dots\dots(i)$$

$$\text{also } a^2 = c^2 - b^2 \dots\dots\dots(ii)$$

$$\text{that is } a^2 = (c - b)(c + b) \dots\dots\dots(iii)$$

$$\begin{aligned} \text{similarly } b^2 &= c^2 - a^2 \\ &= (c + a)(c - a). \end{aligned}$$

Thus, we can find any one side of a right-angled triangle when the other two are given.

Def.—In fig. 4 AB, the side opposite to the right-angle is called the **hypotenuse** (**hyp.**), BC is called the **base** and CA, the **perpendicular** or **altitude** (**alt.**).

Rules. In a right-angled triangle

$$\text{Area} = \frac{1}{2} \times \text{base} \times \text{alt.}$$

$$\text{Hyp.} = \sqrt{(\text{base})^2 + (\text{alt.})^2}.$$

$$\begin{aligned} \text{Base} &= \sqrt{(\text{hyp.})^2 - (\text{alt.})^2} \\ &= \sqrt{(\text{hyp.} + \text{alt.})(\text{hyp.} - \text{alt.})} \end{aligned}$$

$$\begin{aligned} \text{Alt.} &= \sqrt{(\text{hyp.})^2 - (\text{base})^2} \\ &= \sqrt{(\text{hyp.} + \text{base})(\text{hyp.} - \text{base})}. \end{aligned}$$

Note.—In the above rules HYP. BASE and ALT. stand for measure of hypotenuse, base and altitude respectively.

Example. 1.—The base of a right-angled triangle is 60 feet, and the altitude 11 feet; find the hypotenuse.

$$\text{Hyp.} = \sqrt{(60)^2 + (11)^2} = \sqrt{3721} = \underline{61} \text{ feet.}$$

Example. 2.—The hypotenuse of a right-angle triangle is 281 feet, and the base 77 yards; find the altitude.

(N. B.—The given dimensions must be reduced to the *same* denomination before applying the above rules.)

$$77 \text{ yards} = 231 \text{ feet.}$$

$$\text{Alt.} = \sqrt{(281)^2 - (231)^2} = \sqrt{78961 - 53361} = \sqrt{25600} = \underline{160} \text{ feet.}$$

Another method.

$$\text{Alt.} = \sqrt{(281 + 231)(281 - 231)} = \sqrt{512 \times 50} = \sqrt{25600} = \underline{160} \text{ feet.}$$

Note.—The second method is nearly always simpler than the first.

EXERCISES V.

Find the third sides of the following ten right-angled triangles:—

- (i.) base = 133 ; alt. = 156. (ii.) base = 225 ; alt. = 272.
 (iii.) alt. = 68 ; base = 1155. (iv.) hyp. = 1297 ; base = 1295.
 (v.) alt. = 200 ; hyp. = 641. (vi.) hyp. = 905 ; base = 777.
 (vii.) base = 1107 ; alt. = 476. (viii.) hyp. = 1601 ; alt. = 1599.
 (ix.) hyp. = 24649 ; base = 10199. (x.) base = 23660 ; perp. = 23661.

16. Pythagorean Triangles—In the above examples the sides and hypotenuses of the right-angled triangles are integers. Such triangles are frequently called Pythagorean Triangles.

There are methods of writing down *all* the Pythagorean triangles; but they depend on higher mathematics. The student can easily form Pythagorean triangles by attending to the following algebraical relations:—

$$(1.) (p^2 + q^2)^2 = (p^2 - q^2)^2 + (2pq)^2.$$

$$(2.) \left\{ \frac{1}{2}(n^2 + 1) \right\}^2 = \left\{ \frac{1}{2}(n^2 - 1) \right\}^2 + n^2.$$

$$(3.) \left\{ \left(\frac{1}{2}m \right)^2 + 1 \right\}^2 = \left\{ \left(\frac{1}{2}m \right)^2 - 1 \right\}^2 + m^2.$$

Any two whole numbers may be substituted for p and q in (1). Thus putting 4 and 3 for p and q we get 25, 7, and 24 which are the sides of a right-angled triangle.

Any odd number may be substituted for n in (2). Thus, putting 5 for n in (2), we get 13, 12, and 5.

Any even number may be substituted for m in (3). Thus, putting 4 for m in (3) we get 5, 3, and 4.

A list of 15 Pythagorean Triangle, arranged according to hypotenuses up to 100.

3,	4,	5,	9,	40,	41,	48,	55,	73,
5,	12,	13,	28,	45,	53,	13,	84,	85,
8,	15,	17,	11,	60,	61,	36,	77,	85,
7,	24,	25,	16,	63,	65,	39,	80,	89,
12,	35,	37,	33,	56,	65,	65,	72,	97,

PRACTICAL METHODS.

17. When the difference between the two sides containing the right angle of a right-angled triangle is small, the following formula will greatly simplify the process of finding the hypotenuse.

$$c^2 = a^2 + b^2$$

$$= 2ab + (a - b)^2.$$

Ex. $a = 119$, $b = 120$: find c .

$$c^2 = 2 \times 120 \times 119 + (120 - 119)^2$$

$$= 28561$$

$$\therefore c = \underline{169}.$$

If p , q , r be the sides of a right-angled triangle, then will mp , mq , mr be also the sides of a right-angled triangle.

For if $p^2 + q^2 = r^2$

then, also $m^2 p^2 + m^2 q^2 = m^2 r^2$.

Thus, when two given sides of a right-angled triangle have one factor common, this factor may be taken out, and after applying the rules in Art. 15, to the prime numbers, the result may be multiplied by the above common factor to get the final result.

Example.—The sides about the right angle in a right-angled triangle are 1400 feet and 4800 feet respectively. Find the hypotenuse.

Taking out the common factor 200, we get 7 and 24.

$$\text{Now } 7^2 + 24^2 = 169 = 13^2$$

$$\text{Hence, hyp.} = 13 \times 200 = \underline{2600} \text{ feet.}$$

18. The side of an isosceles right-angled triangle is a . Find the hypotenuse.

$$\text{hyp.} = \sqrt{a^2 + a^2} = \sqrt{2a^2} = a\sqrt{2}.$$

Rule. The hypotenuse of an isosceles right-angled triangle is $\sqrt{2}$ times the side.

19. To find the length of the perpendicular drawn from any of the angular points of an equilateral triangle on the opposite side.

Let ABC be an equilateral triangle, and AD the perpendicular from A on BC.

$$\text{Let } AB = BC = CA = a.$$

$$\text{It is evident that } BD = DC = \frac{1}{2}a.$$

$$\begin{aligned} \text{Now } AD^2 &= AB^2 - BD^2 \\ &= a^2 - \left(\frac{1}{2}a\right)^2 \\ &= \frac{3}{4}a^2 \end{aligned}$$

$$\therefore AD = \frac{\sqrt{3}}{2} a.$$

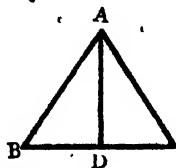


Fig. 5.

Rule. The altitude of an equilateral triangle = $\frac{\sqrt{3}}{2} \times$ the side.

N. B.—The following results must be committed to memory.

$$\sqrt{2} = 1.4142136$$

$$\sqrt{3} = 1.7320508.$$

20 Given the diagonal of a square to find its area.

By § 18, The diagonal = $\sqrt{2} \times$ side

$$\therefore \text{Side} = \frac{\text{diagonal}}{\sqrt{2}},$$

$$\begin{aligned} \text{Hence,} \quad \text{area} &= (\text{side})^2 \\ &= \frac{1}{2}(\text{diagonal})^2 \end{aligned}$$

Rule.—Area of square = $\frac{1}{2} \times$ square of diagonal.

21 **Rhombus:**

Let ABCD be a rhombus, and let AC and BD be its diagonals. It is a well known property of the rhombus that its diagonals bisect each other at right angles and divide the rhombus into four equal triangles.

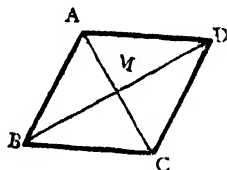


Fig. 6.

We have therefore

$$\begin{aligned} AD^2 &= AM^2 + MD^2 \dots\dots\dots (1) \\ \text{area ABCD} &= 4 \times \text{area AMD} \\ &= 4 \times \frac{1}{2} AM \times MD \\ &= \frac{1}{2} \times 2AM \times 2MD \\ &= \frac{1}{2} \times AC \times BD \dots\dots\dots (2) \end{aligned}$$

Rule.—The side of a rhombus = the square root of the sum of the squares of the semi-diagonals.

Rule.—The area of a rhombus = half the product of the diagonals.

22. Given one side of a right-angled triangle and the sum or difference of the hypotenuse and the other side; to find the hypotenuse and the other side separately.

Let c be the hypotenuse and a and b the sides.

$$\text{Then } c^2 = a^2 + b^2$$

$$\text{or } a^2 = c^2 - b^2$$

$$= (c - b)(c + b)$$

\therefore when a and $c + b$ are given

$$c - b = \frac{a^2}{c + b};$$

and when a and $c - b$ are given

$$c + b = \frac{a^2}{c - b}.$$

Knowing $c + b$ and $c - b$, it is easy to find c and b separately.

$$\text{For } c = \frac{1}{2} \{ (c + b) + (c - b) \} \text{ and } b = \frac{1}{2} \{ (c + b) - (c - b) \}.$$

Example.—If a bamboo, measuring thirty-two cubits and standing upon level ground, be broken in one place by the force of the wind, and the tip of it meet the ground at sixteen cubits, say, at how many cubits from the foot is it broken? (*Lilavati*)

After the fracture, the bamboo assumes the shape of CAB in fig 4.

$$\begin{aligned} \text{Then } b + c &= 32, a = 16; \therefore b - c = \frac{16 \times 16}{32} = 8; \therefore b = \frac{1}{2}(32 - 8) \\ &= \underline{\underline{12 \text{ cubits.}}} \end{aligned}$$

23. Given the hypotenuse and the sum of the sides, find the sides separately.

Here we know c and $a + b$.

$$\begin{aligned}\text{Now } 2c^2 - (a+b)^2 &= 2(a^2 + b^2) - (a+b)^2 \\ &= (a-b)^2\end{aligned}$$

whence $a - b$ is known.

And knowing $a + b$ and $a - b$ it is easy to find a and b separately as in § 22.

Example.—Given $c = 17$ and $a + b = 23$; find a and b .

$$\begin{aligned}(a-b)^2 &= 2c^2 - (a+b)^2 = 2 \times 17^2 - 23^2 \\ &= 49\end{aligned}$$

$$\therefore a - b = 7; a = \frac{1}{2}(23 + 7) = \underline{15}$$

$$b = \frac{1}{2}(23 - 7) = \underline{8}.$$

24. Given the hypotenuse and the difference of the sides; to find the sides separately.

Here we know c and $a - b$

$$\begin{aligned}\text{Now } 2c^2 - (a-b)^2 &= 2(a^2 + b^2) - (a-b)^2 \\ &= (a+b)^2\end{aligned}$$

Thus we find $a + b$. Knowing $a + b$ and $a - b$, we find a and b as in § 23.

Example.—Given $c = 13$, and $a - b = 7$

$$\begin{aligned}\text{Here } (a+b)^2 &= 2 \times 13^2 - 7^2 \\ &= 289\end{aligned}$$

$$\therefore a + b = 17, \text{ and } a = \frac{1}{2}(17 + 7) = \underline{12}$$

$$b = \frac{1}{2}(17 - 7) = \underline{5}.$$

25. By the aid of the property of right-angled triangles established in Euc. I. 47, we can sometimes find exactly the side of a square, the square root of the measure of whose area cannot be found as a commensurable number. Thus if the area of a square be 5 square feet, the value of its side would be $\sqrt{5}$ feet. Now, $\sqrt{5}$ being a surd, its value can be only approximately found. But if we take a line 2 feet in length, and from its extremity erect a straight line 1 foot long at right angles to the first line, the square on the straight line joining the other extremities of these lines will be 5 square feet exactly; since $2^2 + 1^2 = 5$.

Example. To construct a square whose area would be exactly 13 square chains.

Since $13 = 3^2 + 2^2$, the side of the required square will be equal to the hypotenuse of a right-angled triangle whose sides are 3 and 2 chains respectively.

Def. **Commensurable magnitudes** are such as have a common measure. Thus $3\frac{5}{6}$ feet and $6\frac{1}{2}$ feet are commensurable, for they can both be measured exactly by a length of 2 inches.

When magnitudes have no common measure they are said to be **incommensurable**.

Examples of incommensurable magnitudes :—

- (1). The side and diagonal of a square.
- (2). The diameter and circumference of a circle.
- (3). The segments of the divided line in Euc. II. 11.

EXERCISES V.

1. Find the hypotenuse of the right-angled triangles whose sides are respectively :—

- (1) 9 inches and 3 feet 4 inches.
- (2) 11 feet 8 inches and 14 feet 3 inches.

- (3) 37 feet 11 inches and 44 feet.
- (4) 55 feet and 82 feet 5 inches.
2. Find the third side of the right-angled triangles whose hypotenuse and other side are respectively :—
 - (1) 5 feet 5 inches and 4 feet 8 inches.
 - (2) 40 feet 1 inch and 40 feet.
 - (3) 60 feet 5 inches and 53 feet 8 inches.
 - (4) 84 feet 1 inch and 70 feet.
3. The sides of a right-angled triangle are $m^2 - n^2$ and $2mn$. Find the hypotenuse.
4. Two sides of a triangular field are 88 and 110 feet and they include a right-angle; find the area.
5. The hypotenuse of a right-angled triangle is 2275 feet and the base is $\frac{5}{12}$ ths of the altitude: find the area of the triangle.
6. The perimeter of an equilateral triangle is 720 yards: find its altitude.
7. The diagonals of a field, the sides of which are all equal, are 1312 and 1425 links: find its area.
8. Two persons start from the same place at 9 A.M., the one travelling due north by rail at the rate of 35 miles per hour and the other travelling due east by dâk at the rate of 12 miles an hour: how far apart will they be at noon on the same day?
9. Find the length of a diagonal path across a square field containing 20 acres.
10. The hypotenuse and altitude of a right-angled triangle are proportional to the numbers 41 and 40 and the area is 18,000 square yards: find the respective length of the base, the altitude, and the hypotenuse.
11. Find the area in acres of a lozenge-shaped field whose diagonals are 30 and 40 chains.

12. The diagonal of a square is 60 feet ; find its area in square yards.

13. The side of a square is 250 yards ; what is the length of the diagonal ?

14. The width of a house is 60 feet and the height of the ridge above the eaves 16 feet ; required the length of one of the rafters.

15. A ladder is to be placed so as to reach to a window $8\frac{1}{4}$ yds. high, and the foot of the ladder cannot be placed nearer the wall than 9 feet ; what must be the minimum length of the ladder ?

16. Find, correct to three places of decimals, the hypotenuse of the right-angled triangles whose sides are respectively,

(1). 5 ft. and 5 ft. (2). 7 ft. and 9 ft. (3). 27 ft. 3 in. and 18 ft. 2 in. (4). 40 ft. 9 in. and 30 ft. 0 in.

17. A line 97 feet long reaches from the top of a house 72 feet high to the bottom of a house on the opposite side of the street : find the breadth of the street.

18. A ladder 37 feet long reaches to a window 35 feet from the ground : how far is the foot from the base of the building ?

19. The base of an isosceles triangle is 120 feet, and its two sides are each 65 feet ; required the length of the perpendicular from the opposite vertex on the base.

20. The diagonals of a rhombus are 75 and 40 cubits respectively : find the length of a side.

21. A ladder standing upright against a wall 37 feet high was pulled out at the foot 12 feet from the wall ; how far did the top of the ladder fall ?

22. A ladder 25 feet long is standing against a wall. How far must its lower end be drawn away from the wall that its top may be lowered by 1 foot ?

23. A ladder 50 feet long being placed in a street reached a

window 40 feet high on one side; and by turning the ladder over without displacing the foot, it reached another window 48 feet high on the other side. What is the breadth of the street?

24. The diagonal of a rectangle is 32.5 yards, and its area is 252 square yards: what are its dimensions?

25. The area of a rectangle is 3354.12 square feet, and its perimeter is 248.6 feet: find its dimensions.

26. Two ships start from the same port; one sails due south at the rate of 10 miles an hour and the other due west at the rate of $4\frac{7}{8}$ miles an hour: how far apart will they be at the end of 24 hours?

27. The area of a right-angled triangle is 3900 square yards and its base is 936 feet: find the hypotenuse.

28. A rectangular field measures 1184 feet in length and 1113 feet in breadth: what will be the length of a diagonal path across it?

29. The perimeter of a rhombus is 260 feet, and one of the diagonals is 32 feet: find the area.

30. The upright axle of an oilman's machine is placed with its centre $3\frac{1}{2}$ cubits from a wall but the shaft of the axle is 5 cubits in length, measured from the centre: how much of the wall must be removed to allow the shaft to revolve freely.

31. The distance between two stations A and B, as the crow flies, is 65 miles; a third station C is distant 28 miles 7 furlongs from A in a road which is perpendicular to the straight line AB: find the shortest distance between B and C.

32. The base of a right-angled triangle is 4 feet and its area is 1320 square inches; find the hypotenuse.

33. The area of a square is 22.09 square feet. On each side of the square is taken a point 1.2 feet from the nearest corner so that the figure formed by joining these points is a square. What will be the area of this square?

34. The area of a right-angled triangle is 7140 square feet and its hypotenuse 169 feet; find the sides.

35. "In a certain lake swarming with ruddy geese and cranes, the tip of a bud of lotus was seen a span ($\frac{1}{2}$ cubit) above the surface of the water. Forced by the wind, it gradually advanced, and was submerged at a distance of two cubits. Find the depth of the water." (*Lilavati*). [Apply Art. 22]

36. "A snake's hole is at the foot of a pillar 9 cubits high and a peacock is perched on its summit. Seeing a snake, at the distance of thrice the pillar, gliding towards his hole, he pounces obliquely upon him. At how many cubits from the snake's hole do they meet, both proceeding an equal distance?" (*Lilavati*). [Do.]

37. "From a tree a hundred cubits high, an ape descended and went to a pond two hundred cubits distant: while another ape vaulting to some height off the tree, proceeded with velocity diagonally to the same spot. If the space travelled by them be equal, find the height of the leap." (*Lilavati*). [Do.]

38. Apply Euc. I. 47 to construct a square whose area will be exactly one more.

39. The base and altitude of a right-angled triangle are 6.6 and 11.2 metres respectively: find the dimensions and area of the two isosceles triangles into which it can be divided.

40. The area of a square is greater than that of any rectangle with the same perimeter. Prove this. Illustrate this by numerical examples of your own construction.

SECTION III.

TRIANGLES (*generally*).

26. **The area of a triangle.** By Euc I. 41, if a parallelogram and a triangle be on the same base and between the same parallels, the parallelogram is double of the triangle.

Let ABC be a triangle,—acute-angled, right-angled or obtuse angled. As there must at least be two acute angles, let B be one of them.

Fig. 7

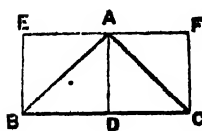


Fig. 8.

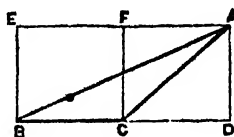
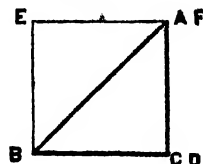


Fig. 9.



From the vertex A , draw AD perpendicular to the base BC (produced if necessary). There are three cases, according as the angle C is acute, right, or obtuse.

If the angle C is acute, D will lie between B and C ,
 if the angle C is right, D will coincide with C ,
 and if the angle C is obtuse, D will lie in BC produced.

In all three cases, draw, through A , EF parallel to BC ; and through B and C draw BE and CF parallel to AD .

Then in each case, the triangle ABC and the parallelogram $BEFC$ are on the same base BC , and between the same parallels BC , EF .

$$\begin{aligned}
 \therefore \text{the triangle ABC} &= \frac{1}{2} \text{ of the parallelogram BEFC} \\
 &= \frac{1}{2} \times \text{rectangle BEFC} \\
 &= \frac{1}{2} \times \text{BC} \times \text{BE} \\
 &= \frac{1}{2} \times \text{BC} \times \text{AD.} \\
 &= \frac{1}{2} \times (\text{base} \times \text{altitude}).
 \end{aligned}$$

RULE.—Area of a triangle $= \frac{1}{2} \times \text{base} \times \text{altitude}$.

$$\text{RULE.}—\text{Alt.} = \frac{2 \times \text{area}}{\text{base}}$$

$$\text{RULE.}—\text{Base} = \frac{2 \times \text{area}}{\text{alt.}}$$

Example 1. The base of a triangle is 20.3 chains and its altitude 39.6 chains : find its area.

$$\text{Area} = \frac{1}{2} \times 20.3 \times 39.6 = 401.94 \text{ square chains.}$$

Example 2. The area of a triangular field is 7 acres, and its altitude 175 yds.; find the base.

$$\text{base} = \frac{2 \times 7 \times 4840}{175} = 387\frac{1}{3} \text{ yds.}$$

27. **Area of an equilateral triangle.** From art. 19, the altitude of an equilateral triangle $= \frac{\sqrt{3}}{2} \times \text{base}$.

$$\text{Hence, area} = \frac{1}{2} \times \text{base} \times \frac{\sqrt{3}}{2} \times \text{base}$$

$$= \frac{\sqrt{3}}{4} \times (\text{base})^2.$$

RULE.—Area of an equilateral triangle.

$$= \frac{\sqrt{3}}{4} \times \text{square of the side.}$$

28. A triangle has six parts or elements, *viz.*, three angles and three sides. The angles of a triangle are denoted by the letters A, B, C; and the sides respectively opposite to them by the letters *a*, *b*, and *c*.

The perimeter of the triangle $ABC = a + b + c$, which is abbreviated by writing $a + b + c = 2s$.

$$\begin{aligned}\text{So that } a + b - c &= a + b + c - 2c \\ &= 2s - 2c \\ &= 2(s - c).\end{aligned}$$

$$\begin{aligned}\text{Similarly, } a - b + c &= 2(s - b), \\ \text{and } -a + b + c &= 2(s - a).\end{aligned}$$

29. In a triangle, the perpendicular from the vertex A on the base $BC = \frac{2}{a} \sqrt{s(s-a)(s-b)(s-c)}$.

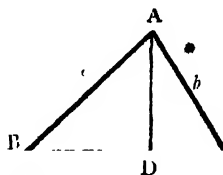
In the triangle ABC , $BC = a$,
 $CA = b$, $AB = c$.

Let AD , the perpendicular from A on $BC = h$.

Since ADB is a right angle,

$$AD^2 = AB^2 - BD^2$$

$$\text{or } h^2 = c^2 - BD^2 \dots \dots \dots (i)$$



But by Euc. II. 13, $AC^2 = AB^2 + BC^2 - 2BC \cdot BD$

$$\text{or } b^2 = c^2 + a^2 - 2a \cdot BD$$

$$\text{or } 2a \cdot BD = c^2 + a^2 - b^2$$

$$\text{or } BD = \frac{c^2 + a^2 - b^2}{2a}.$$

Substituting for BD in (i)

$$\begin{aligned}h^2 &= c^2 - \left(\frac{c^2 + a^2 - b^2}{2a} \right)^2 \\ &= \frac{(2ac)^2 - (c^2 + a^2 - b^2)^2}{4a^2} \\ &= \frac{(2ac + c^2 + a^2 - b^2)(2ac - c^2 - a^2 + b^2)}{4a^2}\end{aligned}$$

$$\begin{aligned}
 &= \frac{\{(c+a)^2 - b^2\} \{b^2 - (c-a)^2\}}{4a^2} \\
 &= \frac{(a+b+c)(c+a-b)(b+c-a)(b-c+a)}{4a^2}
 \end{aligned}$$

$$\begin{aligned}
 \text{Hence } h &= \frac{1}{2a} \sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)} \\
 &= \frac{1}{2a} \sqrt{2s \cdot 2(s-a) \cdot 2(s-b) \cdot 2(s-c)} \\
 &= \frac{2}{a} \sqrt{s(s-a)(s-b)(s-c)}.
 \end{aligned}$$

30. The area of a triangle when the three sides are given.

From art. 26, the area of the triangle $ABC = \frac{1}{2} a \times h$

$$= \frac{1}{2} a \times \frac{2}{a} \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{s(s-a)(s-b)(s-c)}.$$

RULE.—From half the sum of the three sides subtract each side separately, then multiply the half sum and the three remainders together. Then the square root of the product is the area required.

Example.—What is the area of a triangle whose sides are 221, 346, and 525 yards respectively?

$$\begin{array}{r}
 221 \\
 346 \\
 525 \\
 \hline
 2 \overline{) 1092}
 \end{array}$$

$$546 = 2 \times 3 \times 7 \times 13$$

$$546 - 221 = 325 = 5 \times 5 \times 13$$

$$546 - 346 = 200 = 2 \times 10 \times 10$$

$$546 - 525 = 21 = 3 \times 7$$

$$\therefore 546 \times 325 \times 200 \times 21 = 2^3 \times 3^3 \times 5^3 \times 7^3 \times 10^3 \times 13^3$$

$$\text{area} = 2 \times 3 \times 5 \times 7 \times 10 \times 13 = \underline{27300} \text{ square yards.}$$

N. B. The student will observe how the labour of calculation has been greatly lessened by reducing to lowest factors.

30. Hint for simplifying the process—

When all the sides have one factor common, it may be taken out, and after applying the rule in art. 30 to the prime numbers, the result must be multiplied by the *square* of the common factor to obtain the area.

Example.—Let $a = 2600$, $b = 2800$, and $c = 3000$; to find the area.

Taking out the common factor 200, we get 13, 14 and 15.

13	$21 - 13 = 8 = 2 \times 2 \times 2$
14	$21 - 14 = 7 = 7$
15	$21 - 15 = 6 = 3 \times 2$
<u>2 42</u>	
	$21 = 3 \times 7;$

$$\therefore 21 \times 8 \times 7 \times 6 = 2^3 \times 2^3 \times 3^3 \times 7^2$$

$$\text{Hence area} = 200^2 \times 2 \times 2 \times 3 \times 7 = \underline{3360000}$$

EXERCISES VII.

1. Find the areas of the following triangles :-

- (1) $a = 4$ ft. $b = 13$ ft. $c = 15$ ft.
- (2) $a = 39$ yds. $b = 62$ yds. $c = 85$ yds.
- (3) $a = 479$ lks. $b = 169$ lks. $c = 510$ lks.
- (4) $a = 275$ ft. $b = 48$ yds. $c = 221$ ft.

(5) $a = 3140$ ft. $b = 1365$ ft. $c = 2125$ ft.

(6) $a = 4850$ lks. $b = 188200$ lks. $c = 191070$ lks.

II. Find, correct to three places of decimals, the areas of the following triangles:—

(1) $a = 4$, $b = 5$, $c = 7$.

(2) $a = 14$, $b = 25$, $c \approx 35$.

(3) $a = 30$, $b = 97$, $c = 121$.

(4) $a = 141$, $b = 294$, $c = 375$.

32. Theorems of Apollonius.

Def. A line drawn from any angle of a triangle to the middle point of the opposite side is called a Median of the triangle.

Theorem I. In any triangle, the sum of the squares on any two sides is equal to twice the square on half the third side together with twice the square on the median which bisects the third side.

Let ABC be a triangle, and AM the median bisecting BC. Then shall $AB^2 + AC^2 = 2 BM^2 + 2 AM^2$. Draw AD perpendicular to BC.

Then $AB^2 = BM^2 + AM^2 + 2 BM \cdot MD$ (Euc. II. 12) and $AC^2 = MC^2 + AM^2 + 2 CM \cdot MD$.

Hence, by addition, since $BM = MC$, we have $AB^2 + AC^2 = 2 BM^2 + 2 AM^2$.

The above relation enables us to find any of the three medians of a triangle when the three sides are given.

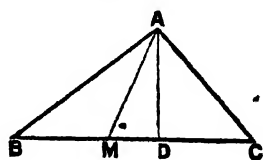
Example. If $a = 14$, $b = 15$, $c = 13$: find AM.

Here $BM = \frac{1}{2} a = 7 \therefore 13^2 + 15^2 = 2 \times 7^2 + 2 \times AM^2$,

or $AM^2 = 169 + 225 - 98 = 296$.

Hence $AM = \sqrt{296} = \underline{17.204}$ nearly.

Fig. 11.



Theorem II. If the base BC of a triangle ABC be divided in M, so that $p \cdot BM = q \cdot MC$, then $p \cdot AB^2 + q \cdot AC^2 = p \cdot BM^2 + q \cdot MC^2 + (p+q) AM^2$.

In the above figure suppose $p \cdot BM = q \cdot MC$.

Making the same construction, we get

$$AB^2 = BM^2 + AM^2 + 2 \cdot BM \cdot MD$$

Multiplying both sides by p

$$p \cdot AB^2 = p \cdot BM^2 + p \cdot AM^2 + 2 p \cdot BM \cdot MD$$

Similarly $q \cdot AC^2 = q \cdot MC^2 + q \cdot AM^2 + 2 q \cdot MC \cdot MD$.

Now, since $p \cdot BM = q \cdot MC$, we have

$$2 p \cdot BM \cdot MD = 2 q \cdot MC \cdot MD$$

Hence, by addition, we get

$$p \cdot AB^2 + q \cdot AC^2 = p \cdot BM^2 + q \cdot MC^2 + (p+q) AM^2$$

Example. If $a = 12$, $b = 15$, $c = 13$, find AM when $2 \cdot BM = CM$ or i.e. when M is the point of trisection of the base nearest to B.

Here $2 \times 13^2 + 1 \times 15^2 = 2 \times 4^2 + 1 \times 8^2 + 3 \cdot AM^2$

$$\therefore AM^2 = \frac{1}{3}(338 + 225 - 32 - 64) = \frac{467}{3} = \frac{467 \times 3}{3} = 1401$$

$$\text{Hence } AM = \frac{\sqrt{1401}}{3} = 12.48 \text{ nearly.}$$

33. To find the segments of the base made by the perpendicular from the opposite angle on it.

We are given the three sides a , b , c .

$$\text{Now } BD^2 + AD^2 = AB^2 = c^2$$

$$CD^2 + AD^2 = AC^2 = b^2$$

$$\therefore BD^2 - CD^2 = c^2 - b^2$$

$$\text{or } (BD + CD)(BD - CD) = c^2 - b^2;$$

$$\text{now } BD + CD = a$$

$$\therefore BD - CD = \frac{c^2 - b^2}{a}$$

Fig. 12.



Hence $BD - CD$ is known.

And we already know $BD + CD$,

whence BD and CD are known separately as in art. 22.

Example. The three sides of a triangle are 13, 14 and 15; find the two parts into which the longest side is divided by the perpendicular from the opposite angle.

$$\text{Here } BD - CD = \frac{14^2 - 13^2}{15} = \frac{27}{15} = \frac{9}{5}; \text{ and } BD + CD = 15$$

$$\therefore BD = \frac{1}{2} \left(15 + \frac{9}{5} \right) = 8.4 \text{ and } CD = \frac{1}{2} \left(15 - \frac{9}{5} \right) = 6.6.$$

In the above figure, the area of the triangle $ABC = \frac{1}{2} BC \times AD$ and the area of the triangle $ABD = \frac{1}{2} BD \times AD$.

Hence the area of the triangle ABD is derived from that of the triangle ABC by dividing by BC and multiplying by BD i.e., multiplying by the factor $\frac{BD}{BC}$. Similarly the area of ACD is

$$= \frac{CD}{BC} \times \text{area of } ABC.$$

Example. Find the area of ABD in the preceding example.

$$\text{Area of } ABC = 84; \therefore \text{area of } ABD = \frac{8.4}{15} \times 84 = 47.04$$

34. A triangle is acute-angled, right-angled or obtuse-angled according as the square of the greatest side is less than, equal to or greater than the sum of the squares of the other two sides. (Euc. I. 47, II. 12, 13.)

Example 1. The sides of a triangle are 7, 10, 15; prove that it is obtuse-angled.

Since $15^2 = 225$, and $7^2 + 10^2 = 149$, $15^2 > 7^2 + 10^2$. Hence, the angle opposite to 15 is an obtuse angle.

Example 2. Classify the following triangles into right-angled,

acute-angled and obtuse-angled. (1) 13, 14, 15. (2) 10, 35, 39.
(3) 7, 15, 20. (4) 65, 72, 97. (5) 13, 15, 18.

EXERCISES VIII.

1. Find the area of an equilateral triangle whose side is 250 feet.
2. Find the area of a triangle whose base is 15 ft. 4 in., and altitude 42 ft. 9 in.
3. Find the area of a triangular field whose sides are 200, 300 and 400 chains.
4. The length of one side of a triangular field is 176 yards and the perpendicular on this side from the opposite angle is 84 yards; find the area of the field.
5. Find the area of an isosceles triangle, whose base is 66 feet, and each of the equal sides 65 feet.
6. The area of a triangle is 694830 square feet, and the base 380 yards; what is the altitude?
7. Find the number of acres in a triangular field whose sides are 3400, 6100 and 7500 links.
8. The perimeter of an equilateral triangle is 72 feet; find the area.
9. The three sides of a triangular rice-field are respectively 260, 510 and 550 chains; find the expense of reaping it at Rs. 3-8 as. per acre.
10. A triangular corn-field, whose base is 609 feet was sold at the rate of 2 as. per square yard and realised Rs. 7612-8 as.; what is the altitude of the triangle?
11. Each of the equal sides of an isosceles triangle is 569 yards,

and the perpendicular from the vertex on the third side is 520 yards : find the area of the triangle.

12. The altitude of an equilateral triangle is 100 feet : find the area.

13. If p represent the altitude of an equilateral triangle, prove that its area is $\frac{\sqrt{3} p^2}{3}$.

14. Find the difference between the area of a triangle whose sides are respectively 45, 85, and 104 feet, and that of an equilateral triangle of the same perimeter.

15. The area of an isosceles triangle is 133980 square feet, and the perpendicular from the vertex on the base 435 feet : find the sides of the triangle.

16. The sides BC, CA, and AB of the triangle ABC are 38, 65, and 87 feet respectively. The perpendicular from P, a point in BC, on the side BA is 10 feet : find the length of the perpendicular from P on CA.

17. In the triangle ABC, $BC = 13$, $CA = 14$, and $AB = 15$. P is a point within the triangle and PD, PE, and PF are perpendiculars from P on BC, CA, and AB. If $PD = 3$, $PE = 4$; find PF.

18. Find the side of an equilateral triangle whose area is 5 acres. Give the answer in feet.

19. A triangular field, 363 yards long and 240 yards in the perpendicular, produces an income of £36 a year. At how much an acre is it let ?

20. In a place where land costs £40 an acre, a triangular field was bought for £300, of which one side measured 302 yards 1 foot 6 inches. What is the height of this triangle in yards ?

21. The sides of a triangle are 340, 65, and 297 feet: find the area in square perches.

22. A triangular field is let at £5-11s.-6½d. an acre for £12. One side is 738 links. Find the perpendicular on this side from the opposite angle to the nearest link.

23. The sides of a triangular field are 2600, 3150, and 2530 feet. Find the area in acres.

24. The sides of a triangle are 102, 104, and 106 feet: find the area in square chains and links.

25. The sides of a triangle are 13, 14 and 15 feet; prove that all its angles are acute and find the perpendicular from the opposite angle on the side of 14 feet.

26. The sides of a triangle are 1200, 1450, and 1650 feet: find the area in square yards.

27. One side of a triangular court is 98 feet, and the perpendicular on it from the opposite angle is 63 feet; required the expense of paving it, at Rs. 1-3-0 per square yard.

28. The paving of a yard in the form of an isosceles triangle costs Rs. 945 at 4 as. per square foot if the base be 40 yards long; find the length of each of the equal sides.

29. Find the side of an equilateral triangle whose area cost as much for paving it at 6 as. per square foot, as fencing the sides did at Rs. 7-8as. a yard.

30. Find the side of a square that shall be equal in area to a triangle whose sides are 1530, 1700, and 2890 feet.

31. Find the area of an isosceles triangle whose base is 2040 feet and each of the equal sides 5151 feet.

32. Find the area of an isosceles triangle whose base is 296 feet, and each of the equal sides 175 feet.

33. Each of the equal sides of an isosceles triangle is 593 yards, and the perpendicular from the vertical angle in the base is 368 yards: find the area of the triangle.

34. The area of an isosceles triangle is 3120 square feet and

the perpendicular drawn from the vertical angle on the base is 80 feet; find all the sides of the triangle.

35. Find the difference between the area of a triangle, whose sides are respectively 520, 730 and 750 feet, and that of an equilateral triangle of the same perimeter.

36. The area of a triangle is 21588.75 square feet and its altitude is 126 feet 3 inches: what is the base?

37. Classify the triangles having the following sides into obtuse-angled, right-angled and acute-angled:—

(1). 301, 900, 901. (2). 8, 123, 125. (3). 116, 181, 225.
(4). 157, 165, 184. (5). 315, 572, 653. (6). 36, 61, 65.

38. Find the lengths of the three medians of the triangle whose sides are 20, 51 and 65 feet respectively.

39. The three sides of a triangle are 25, 101, 114; find the two parts into which the longest side is divided by the perpendicular from the opposite angle.

40. The sides of a triangular field are 217, 404, and 495 feet respectively: find the shortest distances from each corner to the opposite side.

41. Find the area of the triangle formed by joining the middle points of the sides of the triangle whose sides are 89, 99, and 100 feet respectively.

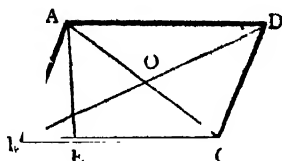
42. The side of a square is 125 feet, a point is taken inside the square which is distant 44 feet and 117 feet respectively from the extremities of a side: find the areas of the four triangles formed by joining the point to the four corners of the square.

SECTION IV. PARALLELOGRAMS.

35. Properties.—(1) The opposite sides and angles of a parallelogram are equal, (2) The diagonal divides it into two triangles which are equal in area, (3). The diagonals bisect each other

Let ABCD be a parallelogram, and let the diagonals bisect each other in O. Draw AE perpendicular from A on BC.

Fig 13.



$$\begin{aligned}\text{Let } BC &= DA = a \\ AB &= CD = c \\ AC &= b \\ BD &= d \\ AE &= h\end{aligned}$$

36 Area of a parallelogram. Since the diagonal AC divides the parallelogram ABCD into two equal triangles ABC, ACD,

$$\begin{aligned}\text{area of ABCD} &= 2 \times \text{area of triangle ABC} \\ &= 2 \times \frac{1}{2} \times BC \times AE \\ &= BC \times AE \\ &= ah.\end{aligned}$$

RULE.—Area of a parallelogram = base \times altitude.

Obs. We can thus find the area of a parallelogram if we are given one side and the perpendicular on it from any point in the opposite side. There are other cases in which we can equally easily find the area of a parallelogram. Take the case in which two adjacent sides and one of the diagonals are given. Here we find the area of one of the triangles by art. 30 and double the result. Thus in fig. 13 if we know AB, BC and CA, we find the

area of the triangle ABC, and double this area to obtain that of the parallelogram.

Again, if we are given the two diagonals AC, BD, and one of the sides AB, we can find the area of ABCD.

For $AO = \frac{1}{2}AC$, $BO = \frac{1}{2}BD$.

Now, in the triangle ABO, we know the three sides, and therefore we can find its area, and the area of the parallelogram is four times that of ABO.

Example 1. In fig. 13, $BC = 726$ feet and $AE = 25$ feet; find the area of ABCD.

Area $= 726 \times 25 = \underline{18150}$ square feet.

Example 2. In fig. 13, $AB = 34$, $BC = 61$, and $BD = 75$: find the area of ABCD.

Now in the triangle ABD, $AB = 34$, $BD = 75$, and $DA = 61$.

\therefore area ABD $= 1020$ (art. 30).

Hence area ABCD $= 2 \times 1020 = \underline{2040}$.

Example 3. In fig. 13, $AC = 50$, $BD = 58$, and $BC = 36$; find the area of ABCD.

Here in the triangle BOC, $BO = 29$, $CO = 25$ and $BC = 36$
 \therefore area BOC $= 360$ (art. 30).

Hence, area ABCD $= 4 \times 360 = \underline{1440}$.

EXERCISES IX.

1. One side of a parallelogram is 525 feet, and the perpendicular from the opposite side is 225 feet: find the area in square yards.

2. A parallelogram, the length of whose sides are 20 and 10 feet, has one diagonal 25 feet long. Find the length of the other diagonal.

3. The area of a parallelogram is 650760 square feet, and a diagonal is 1479 feet. Find the perpendiculars on it from the opposite angles.

4. Find the area of a parallelogram of which one of the diagonals is 525 feet, and the perpendicular on it from the opposite angle 276 feet.

5. Find the area of a parallelogram, the sides being 65 and 119 feet, and one of the diagonals 156 feet.

6. Find the area of a parallelogram, the base being 29 feet and altitude 15 feet.

7. Find the area of a rhombus whose side is 25 feet and altitude 15 feet.

8. The diagonals of a four-sided field the sides of which are all equal, are 6250 and 7500 links : find its area.

9. The side of a rhombus is 53 feet and the longer diagonal 90 feet : find the area and the other diagonal.

10. The two diagonals of a rhombus are 40.8 and 50.6 feet : find the side and the area.

11. The side of a rhombus is 20 and its shorter diagonal is 24. Find the area, and the other diagonal.

12. The area of a rhombus is one acre, and one of the diagonals is 27.5 yards ; find the other diagonal.

Find the altitudes of the parallelograms having the following areas and bases :

13. Area = 1710 sq. yds., base = 180 yds.

14. Area = 7 acres, base = 440 yds.

15. Area = 355680 sq. yds., base = 855 ft.

16. The adjacent sides of a parallelogram are 120 ft. and 180 ft. and its area is half that of a square of the same perimeter : find the two altitudes of the parallelogram.

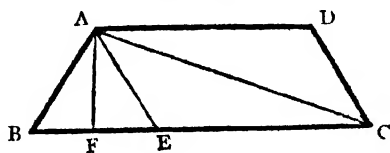
SECTION V.

TRAPEZIUMS.

37. Def. A trapezium is a four-sided figure having only one pair of opposite sides parallel.

Let ABCD be a trapezium having the sides BC, DA parallel.

Join AC. Through A, draw AE parallel to CD, and from A draw AF perpendicular to BC.



Now area of the trapezium ABCD
 = area of triangle ABC + area of triangle ACD
 = $\frac{1}{2} BC \times AF + \frac{1}{2} AD \times AF$. (Since AF is the altitude of both the triangles.)
 = $\frac{1}{2} (BC + AD) \times AF$.

RULE.—Area of a trapezium = half the sum of the parallel sides multiplied by the perpendicular distance between them.

38. Given the lengths of AB, BC, CD, DA to find the length of AF, the perpendicular distance between the parallel sides.

Let $BC = a$, $CD = b$, $DA = c$, and $AB = d$.

Then, $AE = b$, $BE = a - c$.

Let $2p = AB + BE + EA = d + a - c + b$.

Now, by art. 29, $AF = \frac{2}{BE} \sqrt{p(p-AB)(p-BE)(p-AE)}$

$$= \frac{2}{a-c} \sqrt{p(p-d)(p-a+c)(p-b)}$$

$$= \frac{2}{a-c} \sqrt{\frac{(a+b-c+d)}{2} \frac{(a+b-c-d)}{2} \frac{(b+c+d-a)}{2} \frac{(a-b-c+d)}{2}}$$

$$= \frac{1}{2(a-c)} \sqrt{(a+b-c+d)(-a+b-c-d)(-a+b+c+d)(a-b-c+d)}.$$

Hence, area of the trapezium = $\frac{1}{2}$ AF (BC + AD)

$$= \frac{1}{2} \frac{a+c}{a-c} \sqrt{(-a+b+c+d)(a-b-c+d)(a+b-c+d)(a+b-c-d)}$$

Thus, we can find the area of a trapezium when we are given the four sides. .

Obs. By making $c=0$ we get the expression for the area of a triangle, as we should do, for then the side AD becomes evanescent, the points A and D coalesce, and the trapezium becomes a triangle.

Example 1. Find the area of a trapezium whose parallel sides are 35 and 25 feet, the perpendicular distance between them being 28 feet.

$$\text{Area} = \frac{1}{2} (35 + 25) \times 28 = \underline{840} \text{ square feet.}$$

Example 2. The parallel sides of a trapezium are 27 and 41 feet, and the other sides 13 and 15 feet: find the area.

$$\text{Here, } a=41, b=15, c=27, d=13;$$

Substituting in the formula for the area, we get

$$\text{area} = \frac{1}{2} \times \frac{41+27}{41-27} \sqrt{(-41+15+27+13)(41-15-27+13)}$$

$$(41+15-27+13)(41+15-27-13).$$

$$= \frac{1}{2} \times \frac{68}{14} \times \sqrt{14 \times 12 \times 42 \times 16} = \frac{1}{2} \times \frac{68}{14} \times 14 \times 3 \times 8 = \underline{408} \text{ sq. ft.}$$

Another method. In fig. 14, BC = 41, AD = 27, \therefore BE = 14: and AB = 13, AE = 15. Therefore area of the triangle ABE

$$= 84 \text{ sq. ft. (art. 30), and the altitude AF} = \frac{2 \times 84}{14} = 12; \therefore \text{ area}$$

of the parallelogram $AECD = EC \times AF = 27 \times 12 = 324$; adding these two results, area $ABCD = 84 + 324 = 408$ sq. ft.

Note. In fig. 14, the line joining the middle points of AB and AC is parallel to BC and half of BC , and similarly the line joining the middle points of AC and DC is parallel to AD (*i. e.* to BC) and is half of AD . Hence the line joining the middle points of AB , CD is parallel to AD or BC , and is equal to the semi-sum of AD and BC .

EXERCISES X.

1. A quadrilateral field has two of its sides parallel, and they are respectively 1275 and 1325 links in length and the perpendicular distance between them is 524 links: find the area of the field.
2. Find the value of a trapezoidal field the parallel sides of which are 37528 feet and 42472 feet and the perpendicular distance between them 4961 ft., at the rate of Rs. 75 per acre.
3. How many square feet are there in a plank, whose length is 16 ft. 8 in., and the breadths of the two ends $2\frac{3}{4}$ ft. and $2\frac{1}{4}$ ft.?
4. Find the area of a trapezium whose parallel sides are 48 and 60 feet, the other sides being 55 and 65 feet.
5. The parallel sides of a trapezium are 77400 and 140400 feet: the other sides are 25000 and 52000 feet. Find the area in square miles.
6. The area of a trapezium is 475 square feet; the perpendicular distance between the two parallel sides is 19 feet. Find the two parallel sides, their difference being 4 feet.
7. The area of a trapezium is $3\frac{1}{2}$ acres, the sum of the two parallel sides is 297 yards, find the perpendicular distance between them.

8. Calculate the area of a trapezium, the sides of which, taken in order, are 13, 11, 15 and 25, and the second parallel to the fourth.

9. The area of a trapezium is 2.5 acres, and the sum of the parallel sides is 6250 links: find the perpendicular distance between the parallel sides.

10. A trapezium ABCD whose parallel sides AB, DC are 37 and 23 feet respectively is bisected by a line CE drawn from C to a point E in AB: find the length of AE.

11. ABCD is a four-sided figure; AB is parallel to CD; and $AB = BC = DA = 205$ feet; and $CD = 517$ feet: find the area.

12. The sides of a trapezium are 204, 369, 325, 116, yds. and the second is parallel to the fourth; prove that the angle contained by the first two is a right angle and find the area.

13. The end of a house is in the form of a trapezium; the wall in front is 26 feet, and that at the back 34 feet high, and the distance between the walls is 18 feet: find the area of the end.

14. A quadrilateral has two of its sides equal and the other two parallel: the equal sides are 100 feet each, and the parallel sides 600 feet and 760 feet respectively: find the area.

15. The line joining the middle points of the non-parallel sides of a trapezium is 367 yards, and the distance between the parallel sides is 250 yards: find the area.

16. The parallel sides of a trapezium are 8 and 10 metres respectively, and the distance between them is 4 metres. The altitude is divided into four equal parts and lines parallel to the parallel sides are drawn through the points of section. Find the areas of the four trapeziums thus formed.

SECTION VI.

QUADRILATERALS (*generally*).

39. The area of a quadrilateral can be found when any diagonal and the perpendiculars on it from the opposite vertices are given.

Let ABCD be a quadrilateral, and let BE, DF be perpendiculars from B and D respectively on the diagonal AC.

Now, area ABCD = area of the triangle ABC + area of the triangle ACD

$$= \frac{1}{2} AC \cdot BE + \frac{1}{2} AC \cdot DF$$

$$= \frac{1}{2} AC (BE + DF).$$

RULE.—Area of a quadrilateral = half the product of diagonal and the sum of the perpendicular on it from opposite vertices.

Cor. When the diagonals are at right angles to one another it is evident that the area is equal to half the product of the diagonals.

40. If the quadrilateral has a re-entrant angle, that is, if it is concave, the rule in art. 39 requires modification, since one of the diagonals falls without the figure.

Let ABCD be a concave quadrilateral, and let the diagonal BD fall without the figure. Draw AE, CF perpendiculars on BD, from A and C respectively.

Fig. 15.

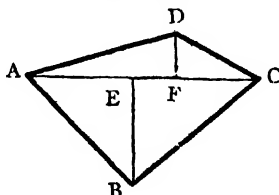
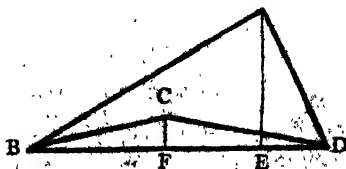


Fig. 16.



$$\begin{aligned}
 \text{Then, area } ABCD &= \text{area } ABD - \text{area } BCD \\
 &= \frac{1}{2} BD \times AE - \frac{1}{2} BD \times CF \\
 &= \frac{1}{2} BD (AE - CF).
 \end{aligned}$$

EXERCISES XI.

1. Calculate the area of park ABCD from the following data :— side AB = 4100 yards, AD = 8400 yards, CD = 10400 yards, BC = 4500 yards, and a road 8500 yards long runs across the park from B to D.
2. One diagonal of a quadrilateral, which lies outside the figure is 135 yards, and the difference of the perpendiculars upon it from the opposite angular points is 25 yards : find the area.
3. One diagonal of a quadrilateral is 640 yards, and the perpendiculars from the opposite angular points are 266 and 134 yards, find the area (1) supposing the diagonal *within*, (2) supposing it *without* the figure?
4. The area of a quadrilateral is 10 acres 2 roods ; and the sum of the perpendiculars on a diagonal from the opposite angles is 154 yards ; find the diagonal.
5. The area of a quadrilateral is 12 acres 2 roods 25 poles : one diagonal is 25 chains : find the sum of the perpendiculars on this diagonal from the two opposite angles.
6. How many square yards are contained in a quadrilateral, one of its diagonals being 60 yards, and the perpendiculars upon it 12.6 and 11.4 yards.
7. How many square yards of paving are there in a quadrangular court whose diagonal is 54 feet, and the perpendiculars on it from the opposite corners 25 and $17\frac{1}{4}$ feet, respectively.

8. Find the area in acres, roods, and poles, of a field ABCD ; $AB = 120$ yards, $BC = 250$ yards, $CD = 100$ yards, $DA = 90$ yards, and the diagonal $AC = 170$ yards.

9. What is the area of a quadrilateral whose diagonals are at right-angles to one another and measure 100 and 125 feet respectively ?

10. Find the area of the quadrilateral ABCD from the following data : $AB = 211$, $BC = 126$, $CD = 155$, $DA = 229$, and the perpendiculars from A on BC and CD, 139 and 166 respectively. Shew that some of the data in this question are superfluous.

11. Find the area of a quadrilateral field ABCD, if $AB = 373$, $CD = 457$, and the diagonal $AC = 331$ feet ; and if the perpendiculars from B and D meet the diagonal in M and N respectively, so that $AN = 262$ and $CM = 178$ feet, and AN greater than AM.

12. One of the diagonals of a quadrilateral field measures 325 links, and the perpendiculars on it from the opposite angles are 125 links and 75 links ; how many square chains does the field contain ?

13. Find the area of the figure PQRS from the following data :— $PQ = 25$, $QR = 33$, $RS = 25$, $SP = 51$ and diagonal $PR = 52$.

14. The sides of a quadrilateral figure are 204, 253, 116, and 231 yards, and the angle contained by the first two is a right angle : find the area.

15. In fig. Euclid I. 47, if ACB be a right-angled triangle, right-angled at C, and AKHC and BEDC squares on the sides AC, BC, find the areas of the quadrilaterals AKEB and AHDB, the sides AC, CB being respectively equal to 48 and 55 feet respectively.

SECTION VII.

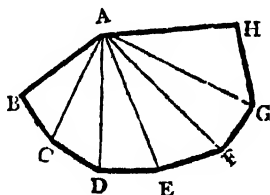
IRREGULAR RECTILINEAL FIGURES.

41. The area of any irregular rectilineal figure can be found by dividing it into triangles and trapeziums, and applying the formulæ in arts. 14, 30 and 38.

We will illustrate this remark by a few examples.

Example 1. Let $A B C D E F G H A$ be a rectilineal figure. Cut it up into triangles by drawing lines from any of the vertices A to the others. Then if we measure the length of $AB, BC, CD, DE, EF, FG, GH, HA, AC, AD, AE, AF, AG$, we can easily find the area of the figure $A B C D E F G H A$.

Fig. 17.

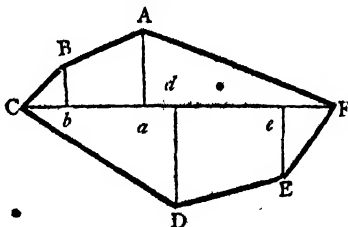


For area $A B C D E F G H A$
 $= \text{area } ABC + \text{area } ACD + \text{area}$

$ADE + \text{area } AEF + \text{area } AFG + \text{area } AGH.$

Example 2. In the rectilineal figure $A B C D E F A$ perpendiculars Aa, Bb, Dd, Ee are drawn from the angular points on the diagonal CF , dividing the figure into the right-angled triangles BCb, AaF, CDd, FEe , and the trapezium $BbaA$ and $DdeE$.

Fig. 18,



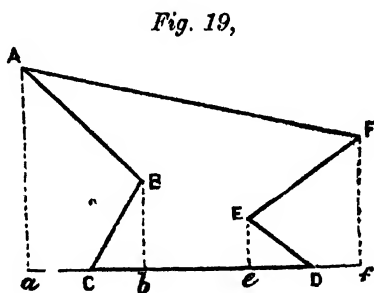
$$\begin{aligned}
 \text{Hence area } ABCDEFA &= CBb + BbaA + AaF + FeE + \\
 &\quad EedD + Ddc \\
 &= \frac{1}{2} Cb \times bB + \\
 &\quad \frac{1}{2}(Bb + Aa)ab + \frac{1}{2} Aa \times aF + \\
 &\quad \frac{1}{2} Fe \times eE + \frac{1}{2} (Dd + Ee) \times de + \\
 &\quad \frac{1}{2} Dd \times Cd.
 \end{aligned}$$

Def.—The lines Aa , Bb , Dd , Ee are called the *offsets* with reference to the line CF .

Sometimes component figures have to be subtracted instead of added as in

Example 3.—In the rectilineal figure $ABCDEFA$, drawing perpendiculars from A , B , E , F on CD produced both ways, we see that area $ABCDEFA = \text{trap.}$

$AafF - \text{trap.}$ $AabB - \text{trap.}$
 $EefF + \text{triangle } BbC + \text{triangle } EeD.$



EXERCISES XII.

1. $ABCDE$ is a five-sided field, calculate its area in acres from the following data:— $BE = 125$ chains, $CE = 136$ chains, the perpendiculars from A and C on BE are 43 and 29 chains respectively and the perpendicular from D on CD is 21 chains.

2. Calculate the area of the field $ABCDE$ from the following data:— $AD = 250$ chains; BP , CQ and ER are perpendiculars on AD and $BP = 110$, $CQ = 95$, and $ER = 24$ chains, and $AP = 10$, and $DQ = 39$ chains.

3. ABCDE is a pentagon; and the angles at A, B, and C are right angles: If $AB = 75$ ft. $BC = 68$ ft. $CD = 40$ ft. $AE = 56$ ft.; find the area of the figure and the length of DE.

4. One side of a quadrilateral field is 250 yards and the perpendiculars from the opposite corners are 135 and 125 yards; the foot of the first perpendicular is 24 yds. from the nearest end of the given side, and the foot of the other perpendicular, 32 yds. from the other end: calculate the area.

5. ABCDEFG is an irregular polygon and Aa, Bb, Dd, Ee, Ff , are perpendiculars on the diagonal CG. If $Gf = 15$ ft., $fe = 35$ ft., $ed = 48$ ft., $cd = 12$ ft., $Aa = 24$ ft., $Bb = 25$ ft., $Ff = 18$ ft., $Ee = 44$ ft., $Dd = 12$ ft., $ab = 10$ ft., $bc = 54$ ft., find the area of the polygon.

6. ABCDEF is a six-sided figure and AP, BQ, DR and ES are perpendiculars on the diagonal CF: $CF = 75$ ft., $AP = 23$ ft., $BQ = 32$ ft., $DR = 42$ ft., $ES = 38$ ft., $CR = 10$ ft., $CQ = 12$ ft., $FP = 18$ ft., $FS = 16$ ft.; calculate the area of ABCDEF.

7. The sides of a field ABCDE, are, respectively, $AB = 65$ yards, $BC = 116$ yards, $CD = 204$ yards, $DE = 252$ yards, $EA = 75$ yards; and the diagonals $BE = 20$ yards, $CE = 120$ yards. Find the area of the field in acres.

8. The length of the diagonal AD of a pentagonal field ABCDE is 524 feet, and the perpendiculars EP, BQ, CR on this diagonal are, respectively 25, 313, 267 feet, these perpendiculars meeting the diagonal at distances 310, 126, and 425 feet respectively from A; find the area of the field.

9. Two perpendiculars AM, CN are drawn to the diagonal BD of the quadrilateral field ABCD, and measure respectively 498 links, and 672 links; also $BM = 76$ and $BN = 513$ links, and the whole

line BD is 999 links : find the area of the field ABCD. Show that some of the data here are superfluous.

10. Find the area of the pentagon ABCDE, from the following data :— $AB = 2100$, $BC = 4500$, $CD = 2200$, $DE = 3400$, $AE = 2400$, and the diagonal $AC = 6000$ and $AD = 5000$.

11. In figure 19 art. 41 find the area of ABCDEF, from the following data :— $Bb = 10$ ft., $Ee = 8$ ft., $Ff = 25$ ft., $Aa = 30$ ft., $aC = 5$ ft., $cb = 6$ ft., $be = 26$ ft., $ed = 13$ ft., $Df = 2$ ft.

12. In fig. 18 art. 41, find the area of ABCDEFA, from the following data :— $CF = 29$ ft., $Cb = 5$ ft., $ba = 10$ ft., $ad = 1$ ft., $be = 21$ ft., $EF = 5$ ft., $CD = 20$ ft., $Bb = 6$ ft., $AB = 26$ ft.

SECTION VIII.

ON ANGLES.

42. **Angular Measurement.**—The only unit of angular measurement referred to in Euclid is the right angle, which is of constant magnitude (ax. 11), and is by its nature the simplest unit angle. Besides this most natural unit, mathematicians use three others, *viz.*, the Degree, the Grade, and the Radian. In the present work we will confine ourselves to the Degree and its sub-divisions.

A *degree* is the ninetieth part of a right angle. The sixtieth part of a degree is called a *minute*, and the sixtieth part of a minute, a *second*. Degrees, minutes, and seconds are symbolised by the marks $^{\circ}$ $'$ $"$, respectively.

Thus 59 degrees 18 minutes and 17 seconds are written $59^{\circ} 18' 17''$.

43. Angles which together make up *one* right angle are said to be **Complementary** to one another; and each angle is said to be complementary of the other. Thus 30° and 60° are complementary angles; so are A and $90^{\circ}-A$.

Angles which together make up *two* right angles are said to be **Supplementary** to one another and each angle is said to be supplementary of the other.

Thus 60° and 120° are supplementary angles; so are A and $180^{\circ}-A$.

44. The three angles of every triangle are together equal to two right angles or 180° and the n angles of a polygon of n sides are together equal to $2(n-2)$ right angles or $180(n-2)$ degrees.

Hence, in any regular polygon of n sides each angle contains $\frac{2(n-2)}{n}$ right angles or $\frac{180(n-2)}{n}$ degrees.

Def. A polygon is said to be **regular** when it has all its sides and all its angles equal.

EXERCISES XII.

1. Find in degrees the angles of (1) an equilateral triangle, (2) a square, (3) a regular pentagon, (4) a regular hexagon, (5) a regular heptagon, (6) a regular octagon, (7) a regular nonagon, (8) a regular decagon, (9) a regular undecagon, (10) a regular duodecagon, (11) a regular quindecagon, (12) a regular icosagon.

2. Express in degrees, minutes, and seconds the following decimals of a right angle :—

(1) .005 (2) .00125 (3) .02 (4) .339 (5) .0001.

3. Express as decimals of a right angle :—

(1) $42^\circ 28' 48''$. (2) $4^\circ 0' 18''$. (3) $44^\circ 16' 48''$

4. Prove that if the sides of a polygon of n sides be produced both ways so as to make a star-shaped figure, the sum of the angles between each alternate pair is $2(n-4)$ right angles.

5. Show that four right angles is the sum of the angles in the regular polygons in each of the following groups :—

(1) Six equilateral triangles (2) two octagons and one square
(3) two hexagons and two equilateral triangles (4) three hexagons
(5) two pentagons and a decagon.

6. The exterior angle of a regular polygon is one-fourth of a right-angle : find the number of sides in the polygon.

7. Given one angle of an isosceles triangle to be 120° : find the other angles.

8. Given two angles of a triangle to be respectively $17^\circ 18' 59''$ and $12^\circ 19' 18''$. find the third angle.

9. How many degrees are there in each of the angles of a regular polygon of 100 sides.

10. One of the acute angles of a right-angled triangle is $43^\circ 37' 57''$: find the value of the other acute angle

11. The vertical angle of an isosceles triangle is $35^\circ 11' 48''$. how many degrees, minutes, and seconds are there in each of the other angles ?

12. Each side of a hexagonal room is 15 feet : Shew that for a mosaic work of the floor, 7 slabs of black marble each having the shape of a regular hexagon, and 12 slabs of white marble each having the shape of an equilateral triangle will be sufficient, the sides of the hexagon and equilateral triangle being each equal to 5 feet.

13. The difference of the two acute angles of a right-angled triangle is $15^\circ 16'$ find the angles.

14. One of the angles of a triangle is half the sum of the other two . find it. •

15. The face of a bee hive is a pattern of regular hexagons. Show that it is mathematically possible.

16. The skin of a fly when examined through a microscope appears to be a beautiful pattern of equal numbers of regular hexagons and equilateral triangles. Draw it and show that it is mathematically correct.

SECTION IX.

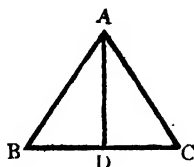
MISCELLANEOUS PROPOSITIONS.

45. Let ABC be an equilateral triangle, and let AD be perpendicular from A on BC.

Then angle ABC = 60° and BAD = 30° .

If $BC = a$, $BD = \frac{1}{2}a$ and $AD = \frac{\sqrt{3}}{2}a$ (art. 19.)

Fig. 20.



RULE.—If in a right-angled triangle one angle = 30° the side opposite to this angle = half the hypotenuse, and third side

= $\frac{\sqrt{3}}{2} \times$ hypotenuse; and of the sides containing the right-angle,

the greater = $\sqrt{3} \times$ the less.

Conversely,

If the hypotenuse of a right angled triangle be double of one of the sides, the angle opposite to this side = 30°

The three following examples, which are worked out in full for the student, contain practical applications of the above two rules.

46. *Example 1.* Two sides of a triangle are 15 ft. and 16 ft. respectively and the included angle is 30° : find the area.

Fig. 21.



Let $AB = 15$, $BC = 16$. Draw AD perpendicular to BC.

Then by art. 45. $AD = \frac{1}{2} AB = 7.5$.

\therefore area = $\frac{1}{2} \times 16 \times 7.5 = 60$ square feet.

RULE.—If the angle included by the sides a, b , of a triangle is 30° , area = $\frac{1}{4} ab$.

47. *Example 2.* In Ex. 1. if the included angle is 60° : find the area.

$$\text{Here, by art. 45, } AD = AB \times \frac{\sqrt{3}}{2} = \frac{15\sqrt{3}}{2}.$$

$$\therefore \text{area} = \frac{1}{2} \times 16 \times 15\sqrt{3} = 60\sqrt{3}.$$

RULE.—If the angle included by the sides a, b , of a triangle is 60° , area = $\frac{\sqrt{3}}{4} ab$.

48. *Example 3.* In Ex. 1, if the included angle is 45° : find the area.

$$\text{Here by art. 18, } AD = AB \times \frac{1}{\sqrt{2}} = \frac{15}{\sqrt{2}}. \therefore \text{area} = \frac{1}{2} \times 16 \times \frac{15}{\sqrt{2}} = 60\sqrt{2}.$$

RULE.—If the angle included by the sides a, b , of a triangle is 45° , area = $\frac{\sqrt{2}}{4} ab$.

49. In Fig. 22. if BD, DC are equal and in the same right line, the triangles BAD, CAD are in equal area. Now, in the triangle BAD, CAD, the sides BD, DA are respectively equal to the sides CD, DA and the contained angles BDA, CDA are supplementary. Hence,

Fig. 22.



RULE.—If two sides of one triangle are respectively equal to the two sides of another triangle and the contained angles are supplementary, the triangles are equal in area.

Thus the area of a triangle in which two sides are 20 ft. and 25 ft. and the included angle 120° = the area of a triangle in which two sides are 20 ft. and 25 ft. and the contained angle 60° . Then proceeding as in Ex. 2. art. 47,

area of either of these triangles $20 \times 25 \times \frac{\sqrt{3}}{4} = 125 \sqrt{3}$ sq. ft.

50. In fig. 23, let $\angle ABC = 60^\circ$, $\angle BAC = 30^\circ$, $\angle ACB = 90^\circ$. Suppose $BC = p$, then $AB = 2p$, $AC = \sqrt{3}p$. (art. 45)

Produce CA to D making $AD = AB = 2p$. Then $\angle BDA = \frac{1}{2} \angle BAC = 15^\circ$.

$$\begin{aligned} \text{Now, } BD &= \sqrt{p^2 + (2 + \sqrt{3})^2 p^2} \\ &= p \sqrt{8 + 4\sqrt{3}} = (\sqrt{6} + \sqrt{2}) p \end{aligned}$$

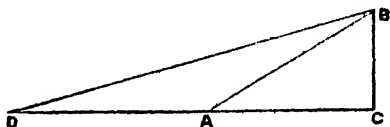
RULE.—If in a right angle one of the acute angles be 15° , the hypotenuse $= (\sqrt{6} + \sqrt{2}) \times$ the side opposite to the angle of 15° , and the third side $= (2 + \sqrt{3}) \times$ the least side.

51. In fig. 23, if $\angle BAC = 45^\circ$, then, $\angle ABC = 45^\circ$, $\angle ACD = 90^\circ$, and $AD = AB$, $\angle ADB = 22\frac{1}{2}^\circ$; suppose $BC = p$, then $AC = p$, $AB = p\sqrt{2}$, $AD = p\sqrt{2}$, and $CD = p + p\sqrt{2} = (\sqrt{2} + 1)p$

$\therefore BD = \sqrt{p^2 + (\sqrt{2} + 1)^2 p^2} = p(\sqrt{4 + 2\sqrt{2}})$. And rules can be deduced as in art. 50.

52. By means of a Theodolite we can measure the angle between the line joining the eye of an observer and any visible object and the horizontal plane through the eye. When the object is above the horizontal plane this angle is called the *angle of elevation*, and when below, the *angle of depression* of the object.

Fig. 23.



Example. At 100 feet distance from the foot of a tree the angle of elevation of the top was found to be 60° ; find the height of the tree.

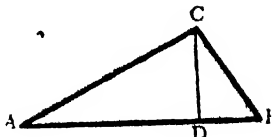
In fig. 4, art 15, $BC = 100$, $\angle B = 60^\circ$

Fig. 24.

Now, by art. 45, $AC = \sqrt{3} \times BC$

. height $= 100 \sqrt{3}$.

53 Let ABC be a right-angled triangle right-angled at C , from C draw CD perpendicular to AB



Then, it is evident from the demonstration in Euc. II 14, that the square on CD is equal to the rectangle AD DB .

Adding the square on DB to both, the rectangle AD , DB together with the square on DB is equal to the sum of the squares on CD , DB .

Hence the rectangle AB . BD is equal to the square on BC . Similarly the rectangle BA . AD is equal to the square on AC .

Def. In fig. 24, AD is called the **projection** of AC on AB , and BD , the projection of BC on AB .

RULE.—In a right-angled triangle, if a perpendicular is drawn from the right-angle on the hypotenuse, the square on the perpendicular is equal to the rectangle contained by the segments of the hypotenuse, and the square on either of the sides forming the right-angle is equal to the rectangle contained by the hypotenuse and the segment of it adjacent to that side (i. e., the projection of that side on the hypotenuse). °

Example.—In fig 24, if $AD = 27$ ft $BD = 3$ ft. : find CD , CB , and AC

$$CD = \sqrt{27 \times 3} = 9 \text{ ft.}, BC = \sqrt{30 \times 3} = 3\sqrt{10} \text{ ft.}; AC = \sqrt{30 \times 27} = 9\sqrt{10} \text{ ft.}$$

53(a) Again referring to fig. 24, we have

$$\frac{1}{AC^2} + \frac{1}{BC^2} = \frac{AC^2 + BC^2}{AC^2 \cdot BC^2}$$

$$\frac{AB^2}{AC^2 \cdot BC^2} = \left(\frac{AB}{AC \cdot BC} \right)^2 = \left(\frac{AB}{2 \cdot \text{area } ABC} \right)^2$$

$$\left(\frac{\frac{1}{2 \cdot \text{area } ABC}}{AB} \right)^2 = \frac{1}{CD^2}$$

RULE. The square of the reciprocal of the perpendicular from the right angle on the hypotenuse of a right-angled triangle = the sum of the squares of the reciprocals of the sides containing the right angle.

Example. In fig. 24, $AC = .3$ ft., $BC = .25$ ft. find CD .

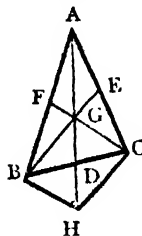
$$\frac{1}{CD^2} = \frac{1}{AC^2} + \frac{1}{BC^2} = \left(\frac{1}{.3} \right)^2 + \left(\frac{1}{.25} \right)^2 = 3^2 + 4^2 = 5^2$$

$$\therefore CD = \frac{1}{5} = .2 \text{ ft.}$$

54. Given the three medians of a triangle : to find the area.

In the triangle ABC , the medians AD , BE , CF meet in one and the same point G . Through C , draw CH parallel to BG , meeting GD produced in H .

Fig. 25.



It is evident from the Geometry of the figure, that $CG = \frac{2}{3} CF$, $BG = CH = \frac{2}{3} BE$, $GH = \frac{1}{3} AD$; so that, when the medians are known, the area of GCH is known.

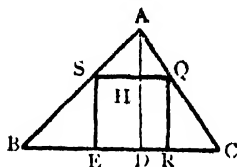
But triangle $CGH = \text{triangle } BGC = \frac{1}{3} \text{ triangle } ABC$. Hence, to find the area of a triangle from the medians, we take two-thirds

of each median and find the area of a triangle having these lengths for its sides by art. 29, and then three times result is the area required.

55. To find the side of a square inscribed in a triangle in terms of the base and altitude of the triangle.

Let SQRE be the square inscribed in the triangle ABC, and let AD be the perpendicular from A on BC. Suppose $BC = a$, $AD = h$, $SQ = SE = HD = QR = x$.

Fig. 26.



Now, area ABC = area ASQ + area SEB + area QRC + area SQRE,

$$\text{or } \frac{1}{2}BC \times AD = \frac{1}{2}SQ \cdot AH + \frac{1}{2}SE \times BE + \frac{1}{2}QR \times RC + SQ \cdot x$$

$$= \frac{1}{2}SQ \cdot AH + \frac{1}{2}SQ (BE + RC) + SQ \cdot x$$

$$\text{i.e., } \frac{1}{2}ah = \frac{1}{2}x(h-x) + \frac{1}{2}x(a-x) + x^2$$

$$\text{or } \frac{1}{2}ah = \frac{1}{2}hx + \frac{1}{2}ax$$

$$\text{or } x(a+x) = ah.$$

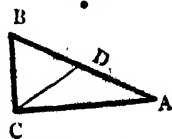
$$\text{Hence } x = \frac{a \times h}{a + h}.$$

RULE.—Side of square inscribed in a triangle = $\frac{\text{base} \times \text{altitude}}{\text{base} + \text{altitude}}$

57. Every right-angled triangle can be divided into two isosceles triangles by a line drawn from the right-angle to the middle point of the hypotenuse.

Let ABC be a right-angled triangle right-angled at C. At C make the angle $ACD = BAC$, then $AD = CD$. Also angle $BCD = ABC$, $\therefore BD = CD$. Hence $AD = CD = BD$, and the triangles ACD and BCD are both isosceles.

Fig. 27.

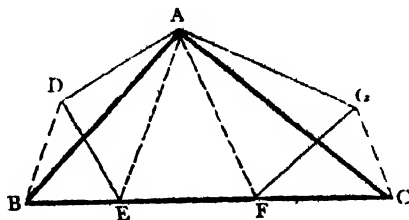


Cer. If $\angle A = 60^\circ$, the triangle ACD is equilateral.

58. The area of an irregular field is often obtained, in practice, by finding the sides of a triangle of the same area as the field.

Let, ADEFG be an irregular rectilinear figure. Join AE, AF.

Fig. 28.



Through D draw DB parallel to AE, and through G draw GC parallel to AF, DB, and GC meeting EF produced in B and C respectively.

Then triangle ABE = triangle ADE and triangle AFC = triangle AGF.

Hence triangle ABC = figure ADEFG.

In practice, the figure ADEFG is drawn to any desired scale and after the simple constructions indicated above the lengths AB, BC, CA are obtained whence the area of ABC (i.e., of ADEFG) follows by art. 29.

EXERCISES XIII.

1. Two sides of a triangle are 105 and 625 feet and the included angle is 30° : find the area.
2. Two sides of a triangle are 200 and 305 feet and they include an angle of 60° : find the area.
3. Two sides of a triangle are 19 and 24 chains, and they include an angle of 45° : find the area.

4 Two sides of a triangular field are 20 and 30 chains, and they include a right angle: find the area.

5. The base of an isosceles triangle is 100 feet, and the vertical angle 120° : find the area.

6. The area of a triangle is 20 square feet, one angle is 45° , and one of the sides which contain this angle is 2.5 feet: find the other sides.

7. Two sides of a triangle are 25 feet and 47 feet and the included angle 45° : find the area.

8. Two sides of a triangle are 157 and 125 feet respectively and the included angle is 30° : find the area.

9. The sides of a triangle are as the numbers 1, 1, $\sqrt{2}$: prove that the angles are as the numbers 1, 1, 2.

10. The angles of a triangle are as the numbers 1, 2, 3; prove that the sides are as the numbers 1, $\sqrt{3}$ and 2

11. Find the area of an isosceles triangle, the vertical angle being 60° and base 100 feet.

12. Find the area of a rhombus whose side is 16 yards, and one of the angles 120° .

13. The sides of a parallelogram are 62.5 and 38.9 chains and the angle between them is 30° : find the area.

14. The angle made by a tight kite-string 100 feet long with the horizon is 30° : find the height of the kite.

15. The sides AB and AC in the triangle ABC are 62 and 97 feet respectively and the median CF = 120 feet: find the area of ABC.

16. The sides AB, AC in the triangle ABC are 25 and 45 feet respectively, and the median AD = 30 feet: find the side BC.

17. The medians BE and CF in the triangle ABC are 114, and 195 feet respectively, and the side BC = 174 feet: find the area of ABC, and the length of the median AD.

18. The medians of a triangle are 105, 156, and 219 feet respectively: find the area of the triangle.

19. The base and altitude of a triangle are 24 and 36 feet respectively: find the side of the square inscribed in the triangle, one of the sides of the square lying on the base of the triangle.

20. The length of the shadow cast by one of the minarets of the Taj Mehal is 76.7 feet, when the angle of elevation of the sun is 60° : find the height of the minaret.

MISCELLANEOUS EXAMPLES.

1. A rectangular pleasure-ground, 351 yards long and 280 yards wide, has in each of its corners a triangular flower-bed with two equal sides of 25 yards each: how much of the ground remains after the flower-beds are deducted?

2. Two steamers start from the same port at the same time, one going N. N. W. at the rate of 6 miles an hour, and the other E. N. E. at the rate of 8 miles an hour; how far are they apart at the end of 8 hours?

3. A verandah is 50 feet long and 4 yards wide and the back wall is 5 feet higher than that in front: find the area of the sloping roof over the verandah.

4. The shadow cast by a man 5 feet 4 inches high is 2 feet: required the height of the Vizianagram tower, which, at the same time casts a shadow of 75 feet. (art. 2).

5. Find the area of the gable end of a house, the breadth of which is 27 feet; the height to the eaves 35 feet, and the height to the apex 45 feet.

6. If $156\frac{1}{2}$ yards of paper, 18 inches wide, are used in papering the walls of a room; how many yards of paper 20 inches wide will be required for the same walls?

7. How many panes of glass, each 9 inches by 7 inches, will be required for 44 windows, each 5 feet by 3 feet 6 inches?

8. What is the excess of a floor 100 feet by 60 feet, above two others, each of half the above dimensions?

9. Along one side of a Court, 23 feet 9 inches square, there is a foot-path 5 feet broad. What will be the cost of laying the rest of it with stones at 4 as. per square yard?

10. A room is 64 feet in circuit, and 16 feet high. How much paper, 2 feet wide, will line it, deducting the door 8 feet by 4 feet, and 4 windows, each 5 feet by 3 feet 6 inches?

11. The base of a right-angled triangle is 300 feet, and the sum of the other two sides is 1,000 feet. What are their lengths?

12. In the quadrilateral field ABCD, on account of obstructions, there could only be taken the following measurements:—BC = 265 yds., AD = 221 yds., the diagonal AC = 378 yds., and the two distances of the feet of the perpendiculars BE, DF from the extremities of the diagonal, viz. AE = 10 yds. and CF = 236 yds. Find the area in acres.

13. A rectangular room is twice as high as broad, and twice as long as high: find its dimensions, having given that the cost of asphaltting the floor came to Rs. 675 at 6 as. per square yard.

Find also the cost of plastering the walls and ceiling at Re. 1-4as.

per 100 sq. ft. deductiong one-fifth of the area of the walls for doors, &c.

14. The hypotenuse of a right-angled triangle is 2125 feet, and the base is $\frac{3}{7}$ ths of the altitude : find the area of the triangle.

15. The diagonal of a square court-yard is 66 feet : find the cost of paving the yard at Rs. 3 - 12as. per 25 square yards.

16. The sides of a triangle are 3465, 3344, and 3047 feet ; find the area of the triangle in square yards.

17. In a quadrilateral ABCD, $AB = 175$, $BC = 527$, $CD = 261$, $DA = 357$, the diagonal $AC = 600$. Find the area.

18. What length must be cut off a straight plank $1\frac{1}{2}$ feet broad in order that its surface may contain 10 square yards ?

19. What must be the side of an equilateral triangle, so that its area may be equal to that of a square of which the diagonal is 180 feet ?

20. A quadrilateral space ABCD. Find area in acres and decimals of acres. $AB = 297$ yards, $BC = 304$ yards, $CD = 250$ yards, $DA = 225$ yards, $AC = 425$ yards.

21. Find what length of wall-paper 27 inches wide will be required for a room 20 feet long, 16 broad, and $10\frac{1}{2}$ feet high. In it are two windows 6 feet \times 4 feet ; a door 7 feet \times 4 feet ; and a fire-place 4 feet \times 3 feet 6 inches.

22. The sides of a quadrilateral taken in order are 8, 8, 7, 5 feet, respectively ; and the angle contained by the first two sides is 60° , find the area.

23. There is a garden 140 feet long, and 120 feet broad : and a gravel walk is to be made of an equal width half round it ; on the inside, the breadth of the walk being 10 feet : find its area.

24. Two of the four hedges of a field are parallel and 1000 yards and 936 yards long, respectively. A man, standing midway between these parallel hedges, observed that a horse he was lunging with 25 yards of rope, in crossing the shortest line from his station to either parallel hedges bisected it. Required the area of the field in acres.

25. The sides of a triangle are 20, 30, 40 chains. Find its area in square miles.

26. Two square fields jointly contain 10 acres, and the side of one is three-fourths as long as that of the other. How many acres in each?

27. The diagonals of a rhombus are 60 feet and 45 feet, respectively; find the area; find also the length of a side, and the height of the rhombus.

28. How much paper $\frac{5}{13}$ of a yard broad will be required to paper a room which is 22 feet by 20 and 13 feet high?

29. The side of a square is 100 feet; a point is taken inside the square which is distant 60 feet and 80 feet respectively from the two ends of a side: find the areas of the four triangles formed by joining the point to the four corners of the square.

30. How many square metres are there in a trapezium, the parallel sides of which are 157.6 metres and 94 metres, and the perpendicular distance between them 72 metres?

31. How long will it take to walk round a field containing 13 acres 1089 square yards in the form of a square, at the rate of $2\frac{1}{4}$ miles an hour?

32. The area of a rhombus is 354144 square feet, and one diagonal is 672 feet; find the other diagonal, find also the length of a side, and the height of the rhombus.

33. Find, by duodecimals, the area of a rectangle which measures 9 feet 9 inches 10 twelfths by 4 feet 6 inches 7 twelfths.

34. Find the area of a triangle whose sides are 14.8 yards, 15.3 yards, and 17.5 yards.

35. Find the expense, of carpeting a room 26 feet long and 18 feet broad, with carpet 27 inches wide, at 4s. 8d. per yard.

36. On a map drawn to the scale of $\frac{1}{10000}$ the sides of a rectangular field are .65 and .72 inches. Find the area of the field in acres, and the length of the diagonal in yards.

37. Two square rooms, one having its dimensions twice each way of the other, are of equal height, and cost together £8-6s 8d to paper the walls at 6d. per square yard and £100 to carpet the floors at 4s. per sq. ft. : find the height and the lengths.

38. The sides of a rectangular lawn are 45 yards 1 foot, and 37 yards 1 foot 6 inches respectively ; how many additional square feet of turf would it take to convert it into a square lawn with one of the diagonals as diameter ?

39. The area of the two side-walls of a rectangular room is 806 square feet, the area of the two end-walls 546, and the area of the floor 651 ; find the dimensions of the room.

40. In a triangle the perpendicular on the base from the vertex is 5 feet, and the perpendiculars from the foot of the first perpendicular on the sides are 4 feet and 3 feet respectively ; find the area of the triangle.

41. Find the length of a zigzag road ascending by a gradient of 1 in. in 8 to the top of the Patesnath Hills which is 4,500 feet high.

42. The sides of a triangle are 15, 14 and 13 feet; find the area in square links.

43. One side of a right-angled triangle is 3.925; the difference between the hypotenuse and the other side is 625 feet; find the hypotenuse and the other side.

44. The side of a square garden is 50 yards, and on its outside is a walk of an equal width of 5 feet. Find the cost of planting the border of the walks with box-wood, at 2 pies per running foot.

45. Find the area in acres of a field ABCD. AD = 390 yards, BC = 370 yards, AC = 400 yards, AB = 130 and CD = 250 yards.

46. A rectangular garden measures 36 yards 2 feet 9 inches, by 24 yards 1 foot 10½ inches. Find the difference in area between a path 4½ feet wide round the outside, and one 6 feet wide round the inside.

47. Two sides of a parallelogram are 10.62 and 15.35 feet and the angle between them is 30°: find the area.

48. A square field of grain containing 10 acres is to be cut by a reaper round and round. The cut of the reaper is 5 feet. How many rounds must the reaper take to cut one-fourths of the field?

49. The base of a triangular field is 1,210 yards, and the height is 496 yards; the field is let for £248 a year: find at what price per acre the field is let.

50. What length of matting, ¾ of a yard wide, will be required for a room 15 feet 8 inches long by 11 feet 3 inches wide; and what will be its cost at 6 annas per yard?

51. A rectangular area is a mile long and 157½ yards broad:

find the length of a fence running from a corner to the shorter side that will cut off $\frac{1}{4}$ acres of ground.

52. The area of a rhombus is 120,000 sq. feet, and the side 400 feet: find the diagonals.

53. The parallel sides of a trapezium are 55 and 88 feet, and the other sides are 25 and 52 feet: find the area.

54. A railway platform has two of its opposite sides parallel, and its other two sides equal, the parallel sides are 100 and 120 feet respectively, and the equal sides are 15 feet each. Find its area.

55. The sides of a triangular field are 350, 440 and 750 yards; the field is let for £26-5 s. a year; find at what price per acre the field is let.

56. The three sides of a triangle were 800, 500, and 1,100 links. By some mistake the third side was put down as 500 instead of 1,100. What error would that mistake occasion in the computed area?

57. The side of a rhombus is 20 feet, and its shorter diagonal is $\frac{3}{4}$ ths of the longer one; find its area.

58. From a point within an equilateral triangle, perpendiculars are drawn to the three sides, and are 8, 10, and 12 feet, respectively. Find the side, and the area of the triangle.

59. The sides of a triangle are 13,000, 37,000, and 40,000 feet respectively: find the ratio of its area to that of an equilateral triangle of the same perimeter.

60. A triangular field whose sides measure 375, 300 and 225 yards, is sold for £8,500. Find the price per acre.

61. The area of a square is 22.3. Find the side of a square of half the size.

62. The side of a rhombus is 36 feet, and one of its diagonals is 18 feet, find the other diagonal and the area of the figure.

63. The paving of a triangular court came to £ 94½ at 15 *d*. per square foot. If one of the sides be 24 yards long, find the length of the other two equal sides.

64. How many planks 10 feet long, and 8 inches broad, will be required for the floor of a room whose length is 30 yards and breadth 12 yards?

65. The area of an equilateral triangle is 1943.737 square feet, find its side.

66. The diagonal of a square court is 300 feet, find its area in square yards.

67. The sides of a plot of ground are 490, 300, 400 and 300 yards respectively, and one of the angles adjacent to the greatest side is 90°. Find the area in acres.

68. One diagonal of a quadrilateral, which lies outside the figure, is 140, and the difference of the perpendiculars upon it is 32; find its area.

69. What is the area of a triangle whose sides are 165, 220 and 275 feet? Find the answer in roods and perches.

70. The area of a rectangular court-yard is 360 square yards, and its sides are in the ratio of 1 to 4. A pavement of uniform width runs along two adjacent sides of the court-yard, and occupies half its area. Show that the width of the pavement is 4 yds.

71. The sides of a triangle, of which the perimeter measures 462 feet, are in the ratio of 6, 7 and 8. Find its area.

72. What is the difference between the superficial contents of a floor 28 feet long and 20 broad, and that of two others of only half its dimensions?

73. What must be the side of an equilateral triangle, so that its area may be equal to that of a square, of which the diagonal is 120 feet?

74. The sides of a triangle are respectively 1500, 1700, and 2300 links. Required the area in acres, roods and perches.

75. The rent of a square field at £ 2-14 s.-6 d. per acre, amounts to £ 27-5 s. Find the cost of putting a paling round the field at 9 pence per yard.

76. The length of a railway is $47\frac{1}{2}$ miles, and the average breadth of land required for its formation 57 yards; what will be the amount of purchase of the land at £50 per acre?

77. One of the parallel sides of a trapezium is one foot longer than the other, the breadth is one foot, and the area 216 square inches. Required each of the parallels.

78. A field whose three sides are equal cost Rs. 55-6-9 turfing, at the rate of 5 annas per 100 square feet; find the length of one of its sides.

79. Find the cost of painting the gable end of a house at 1s 9d per square yard: the breadth being 27 feet, the distance of the eaves from the ground 33 feet, and the perpendicular height of the roof 12 feet.

80. The sides of a triangle are 25, 39, and 56 feet, respectively: find the area of the two triangles into which it is divided by the perpendicular from the angle opposite the largest side on that side.

81. The whole surface of a cube is 3750 sq. ft.: find the length of the edge of the cube.

82. What length of matting, $\frac{1}{4}$ of a yard wide, will be required for a room 15 feet 8 inches long by 11 feet 3 inches wide; and what will be its cost at 6 annas per yard?

83. A rectangular enclosure is 120 feet long and 70 feet broad; a walk of uniform width is to be made round the *outside* of it equal in area to the enclosure; find the width of the walk.

84. A room measures 28 feet by 16 feet. In the centre is a Turkey carpet 24 feet by 12 feet; how much oil-cloth would be required to cover the remainder of the floor, supposing the oil-cloth to be 20 inches wide?

85. Find how many planks, 12 feet 6 inches long by $9\frac{1}{2}$ inches wide, will be required to floor a room 50 feet by 19 feet.

86. The area of a quadrilateral with two of its sides parallel is 15 acres, and the sum of the parallel sides 275 yards; find the perpendicular distance between the parallel sides.

87. A ladder 25 feet in length rests against a wall with its top 24 feet from the ground. It is tied to the bottom of the wall by a string fastened to the middle point of the wall. Find the length of the string, and prove that it cannot prevent the ladder from slipping.

88. A rectangular field is 1320 yards long, and 1155 yards wide; find its area in acres. Also find the areas of the portions into which it is divided by a straight line drawn from the middle point of one side to one of the opposite corners.

89. The breadth of a room is two-thirds of its length and three-halves of its height, the area of the four walls 1920 square feet; find the dimensions of the room.

90. The sides of a triangle are 29, 35, and 48 feet respectively,

find the areas of the two triangles into which it is divided by the perpendicular from the opposite angle on the largest side.

91. A room whose height is 10 feet and length twice its breadth, takes 160 yards of paper 2 feet wide for its walls : find how much carpet $\frac{3}{4}$ yd wide it will require.

92. If each side of a certain square were increased by 8 feet, the area of the new square would be 256 square feet more than that of the first square : find the side of the second square.

93. A rectangular orchard, 231 feet long and 160 feet wide, has a path 4 feet wide on the shorter sides, and a path $4\frac{1}{2}$ feet wide on the longer sides, and also a gravel path 5 feet wide across it each way : how much ground is left for cultivation ?

94. The sides of a triangle are 13332, 7161, and 16335 feet : find the area in acres, roods and perches.

95. The diagonal of a rectangle, the breadth of which is 100 feet is 1252 feet ; find the side of a square of equal area.

96. The length of a room is twice the breadth ; the cost of white-washing the ceiling at 9 as. per square yard came to Rs. 10-2 as. and the cost of painting the walls, after making a deduction of one-sixth of the area of the walls for doors &c., was Rs. 140-10 as. at 5 as. per square foot : find the height of the wall.

97. The sides of a triangle are 28 ft. and 65 ft. and the included angle 120° : find the area.

98. The sides of a triangle are 125 yds. and 75 yds. and the included angle 150° : find the area. [apply art. 49.]

99. Two angles of a triangle are 30° and 60° and the side adjacent to both is 100 ft. : find the area. .

100. Two sides containing the obtuse angle of an obtuse-angled triangle are 11 and 25 yds. respectively, and the area is 132 sq. yds. ; find the third side.

101. Three consecutive sides of a quadrilateral are 50, 50 and 48 feet respectively. The angle between the first two is 60° , and the opposite angle is 90° : find the area. •

102. The area of a triangle is 640 sq. ft., one angle is 30° , and one of the sides containing this angle is 32 ft. : find the other sides.

103. The diagonals of a quadrilateral intersect one another at right angles and their lengths are 87 ft. and 25 ft. : find the area.

104. The sides of a parallelogram are 40 and 50 ft. in length and contain an angle of 60° : find the area, and the diagonals.

105. The angle of elevation of the top of Asoka's pillar at Allahabad from a place 295 inches from the base of the pillar is 60° : find the height of the pillar,

106. The sides of a triangle are 17, 24, 29 ; calculate the distance from the middle point of the shortest side to the opposite vertex.

107. The sides containing the right angle of a right-angled triangle being 13, 5 and 60, calculate the areas of the two triangles into which the whole is divided by the perpendicular from the right angle on the hypotenuse. •

108. Given the area of a triangle 4 acres ; its base 500 links, and one side 1,000 links, to find the other side.

109. Given an irregular heptagon, show how to reduce it to a triangle of equal area.

110. A trapezium with parallel sides of lengths as 3:4 is cut from a rectangle $12' \times 2'$, so as to have an area of $\frac{1}{3}$ of the latter. Find the lengths of the parallel sides.

111. Two straight lines which cross one another meet a canal at angles of 30° and 60° , respectively. If it be 3 miles by the longer of the two roads from the crossing to the canal; find the distance by the shorter. If there be a footpath which goes the shortest way to the canal; find the distance by it.

112. A person observes the angle of elevation of a tower to be 30° , and on approaching 60 feet, he finds the elevation to be 60° ; find the height of the tower.

113. A quadrilateral has three sides, each 120 yds. in length, and the fourth side is 160 yds. Find the area in acres if one angle adjacent to the greatest side is a right-angle. (Cal. Univ.)

114. The sides of a triangle are 50, 78, and 80, find the area, and the lengths of the three perpendiculars from the angular points on the opposite sides.

115. The angles of elevation of the top of a hill from two consecutive mile-stones on a road running straight to the base of the hill are 30° and 45° respectively; find the height of the hill.

116. ABCDEF is a six sided figure; BK, CL, EM, and FN are perpendiculars on AD. If $BK = 3$ ft., $CL = 4$ ft., $EM = 4$ yds., $FN = 5.1$ ft., $AK = 8$ ft., $KL = 5$ yds. $LD = 60$ inches, $AN = 14$ ft., $NM = 13$ ft.; find the area of the figure in square feet.

117. ABCD is a quadrilateral whose area is 8575 square yards, B is a right angle, BL is perpendicular to AC, AL = 90 yards, CL = 40 yards: find the area CAD.

Also, if P is the middle point of CD, and PQ is parallel to CA, find the length of PQ. (Cal. Univ).

118. Two sides of a triangle are 9 and 12 feet respectively, and the angle contained by them is equal to the sum of the sum of the other two: find the third side and the area (Cal. Univ).

119. What regular polygon has each of its angles equal to nine-tenths of two right angles. (Cal. Univ).

120. The sides of a triangular field are 175 yards, 200 yards, and 250 yards in length respectively. Find the area in acres, roods, and perches. (Cal. Univ).

121. After measuring 15 apparent miles of road, a man discovers that his chain is 2 inches too long. Find the error of measurement. What would be the error in measuring a square field which apparently contains 12 acres 2 roods? (Cal. Univ).

122. A person at a place A cannot pass directly to another place B, but can do so in the following manner:—He first walks due East 100 feet, then North-East 120 feet, then North-West 150 feet, and finally by going South-West 60 feet, reaches B. Find the distance between A and B. (Cal. Univ).

123. The sides AB, BC, CD of a quadrilateral ABCD are 10, 15, and 16 respectively, and the angles ABC and BCD are 135° and 90° respectively; find the length of AD correct to one decimal place. (Cal. Univ).

124. The two parallel sides of trapezium are 12 and 6, and the other two are 7 and 8; find the lengths of the diagonals. (Cal. Univ).

125. If a plane figure be entirely made up of an unlimited number of equal equiangular and equilateral rectilineal figures, prove that these figures can only be triangles, squares, or hexagons.

126. Prove that of all equiangular and equilateral rectilineal figures of which an unlimited number can entirely cover a plane superficies, the hexagon with a given perimeter encloses the greatest area.

PART SECOND.

[***This part presupposes a knowledge of the first Four Books of Euclid.]
SECTION I.

TANGENTS AND CHORDS OF A CIRCLE.

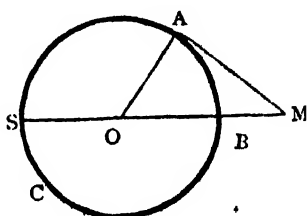
59. Length of Tangent.

Let MA, be the tangent from any point M to the circle ABC, whose centre is O. Join AO.

Now, the angle OAM being a right angle Euc. III. 19.

we have $MA^2 = MO^2 - OA^2$
 $= (MO + OA)(MO - OA).$

Fig. 29.



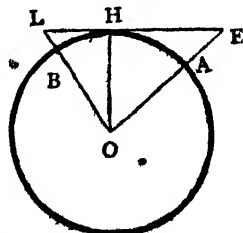
RULE. The length of the tangent from any point to a circle is the square root of the product of the sum and difference of the distance of that point from the centre and the radius of the circle.

60. Terrestrial Horizon.

The earth is approximately a sphere of 4,000 miles radius.

If from a height L (h feet from the surface of the earth) a series of tangents like HL be drawn, they will touch the earth in a circle, called the Terrestrial Horizon.

Fig. 30.



61. Distance of the Horizon at sea.

In figure 30, let $LB = h$ feet $= \frac{h}{5280}$ miles

$$\therefore LO = \left(4,000 + \frac{h}{5280}\right) \text{ miles}$$

$$\text{Now } LH^2 = LO^2 - HO^2 = (LO + HO)(LO - HO)$$

$$\begin{aligned} &= \left(4,000 + \frac{h}{5280} + 4,000\right) \\ &\times \left(4,000 + \frac{h}{5280} - 4,000\right) \\ &= \left(8,000 + \frac{h}{5280}\right) \frac{h}{5280} \\ &= \frac{8,000 \times h}{5280} + \left(\frac{h}{5280}\right)^2 \\ &= \frac{800 \times h}{528} \text{ nearly} \end{aligned}$$

$$\text{since } \left(\frac{h}{5280}\right)^2 \text{ is very small compared to } \frac{800h}{528}$$

$$\begin{aligned} &= \frac{(3 \times 264 + 8) h}{2 \times 264} \\ &= \frac{3h}{2} + \frac{8h}{2 \times 264} \end{aligned}$$

Hence $LH = \sqrt{\frac{3h}{2}}$ miles nearly. (Since $\frac{8h}{2 \times 264}$ is very small compared to $\frac{3h}{2}$).

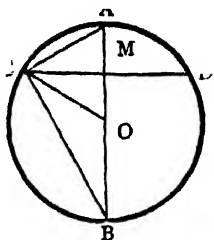
RULE. The distance of Sea Horizon from a height of h feet = $\sqrt{\frac{3h}{2}}$ miles.

Obs. In fig. 30, the distance LE between two spectators who can just see each other on the masts BL and AE of heights h

and h' respectively = $\left(\sqrt{\frac{3h}{2}} + \sqrt{\frac{3h'}{2}}\right)$ miles.

62. Let CD be any chord of the circle $ACBD$, and let O be the centre of the circle. From O draw OM perpendicular to CD and produce OM both ways to meet the circumference in A and B . Join CA , CO , CB .

Fig. 31.



It is evident that M is the middle point of the chord CD (Euc. III. 3) and A , the middle point of the arc CAD , and ACB is a right angle (Euc. III. 31).

Def. CD is called chord of the arc CAD , and AM height of the arc, and CA , chord of half the arc.

63. In fig. 31, if we describe a circle about the triangle CMB , AC will be a tangent to this circle; since ACB is a right angle (Euc. III. 19).

Therefore $CA^2 = AM \cdot AB$, (Euc. III. 36).

Again, by Euc. III. 35, the rectangle $AM \cdot MB = CM \cdot MD$
 $= CM^2$.

These two results, viz.

$$CA^2 = AM \cdot AB \dots\dots\dots I$$

$$\text{and } CM^2 = AM \cdot MB \dots\dots\dots II$$

are very useful in solving questions on the chords of a circle.

[They were deduced by another method in art 53.]

64. Given the height (AM) of the arc (CAD) and the chord of half the arc (CA) to find the diameter (AB).

$$\text{Since } AM \cdot AB = CA^2 \dots\dots\dots I.$$

$$\therefore AB = \frac{CA^2}{AM}.$$

65. Given the height (AM) of the arc (CAD) and the diameter (AB), to find the chord of half the arc.

Since $AM \cdot AB = CA^2$ I

$$CA = \sqrt{AM \cdot AB}.$$

66. Given the diameter (AB) and the chord of half the arc (CA); to find the height (AM) of the arc.

Since $AM \cdot AB = CA^2$ I

$$AM = \frac{CA^2}{AB}.$$

67. Given the chord (CD) of the arc (CAD) and the height (AM) of the arc, to find the diameter (AB).

Since $CM = \frac{1}{2} CD$, and $CM^2 = AM \cdot MB$ II

$MB = \frac{CM^2}{AM}$ whence MB is known, and $AB = AM + MB$.

68. Given the height (AM) of the arc (CAD) and the diameter AB; to find the chord (CD) of the arc.

$CA^2 = AM \cdot AB$ whence CA is known, and $CM^2 = CA^2 - AM^2$ whence CM is known and $CD = 2 \cdot CM$.

69. Given the chord (CD) of the arc (CAD) and the diameter (AB): to find the height (AM) of the arc.

$CM = \frac{1}{2} CD$; and $CO = \frac{1}{2} AB$,

and $MO^2 = CO^2 - CM^2$

whence MO is known.

and $AM = AO - MO = \frac{1}{2} AB - MO$.

70. Given the chord (CD) of the arc CAD and the diameter AB; to find the chord of half the arc CA.

$$CM = \frac{1}{2} CD$$

$$CO = \frac{1}{2} AB$$

and $MO^2 = CO^2 - CM^2$ whence CM is known.

Again $AM = AO - MO$

and $CA^2 = AM^2 + CM^2$ whence CA is known.

71. FORMULÆ.

If $2C$ = chord of the whole arc,

H = chord of half the arc,

a = height of the arc,

d = diameter of the circle,

then

$$H = \sqrt{C^2 + a^2} \dots\dots\dots (i)$$

$$d = \frac{H^2}{a} \dots\dots\dots (ii)$$

$$H = \sqrt{ad} \dots\dots\dots (iii)$$

$$a = \frac{H^2}{d} \dots\dots\dots (iv)$$

$$d = \frac{C^2}{a} + a = \frac{C^2 + a^2}{a} \dots\dots\dots (v)$$

72. Distance of chord from centre.

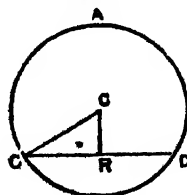
Fig. 32.

Let OR be the perpendicular from the centre O on CD ; then $CR = RD$ (Euc. III. 3).

Now, $OR^2 = CO^2 - CR^2$

$$= (CO + CR)(CO - CR)$$

Whence OR is known.



RULE. Distance of chord of length c from the centre

of a circle of radius $r = \sqrt{r^2 - \left(\frac{c}{2}\right)^2} = \sqrt{\left(r + \frac{c}{2}\right)\left(r - \frac{c}{2}\right)}.$

Obs. There can be drawn an indefinitely large number of chords of the same length in a circle; and there cannot be drawn more than two parallel chords of the same length in a circle. The length of the chords must, of course, be less than that of the diameter.

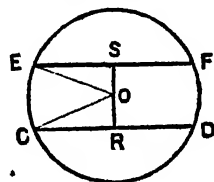
73. Distance between parallel chords of given lengths.

Let the lengths of the chords CD and EF be c and c' , and let the radius be r .

First, let the chords be on opposite sides of the centre.

Draw OR, OS perpendiculars on CD, EF respectively. Then SO, OR are in one and the same straight line.

Fig. 33.



$$\text{Now OR} = \sqrt{r^2 - \left(\frac{c}{2}\right)^2}, \text{ and OS} = \sqrt{r^2 - \left(\frac{c'}{2}\right)^2}$$

$$\text{Hence SR} = \sqrt{r^2 - \left(\frac{c}{2}\right)^2} + \sqrt{r^2 - \left(\frac{c'}{2}\right)^2}.$$

Secondly, when the chords are on the same side of the centre, it is evident that we must take the difference of OR and OS instead of their sum.

74. From Euc. III, 11 and 12 we obtain the following theorem on the relative positions of two circles:—

Two circles will (1) be wholly outside each other, (2) touch externally, (3) intersect in two points, (4) touch internally, or (5) one of them will be wholly within the other, according as the distance between their centres is (1) greater than the sum of their radii, (2) equal to the sum of their radii, (3) less than the sum and greater than the difference of their radii, (4) equal to the difference of their radii, or (5) less than the difference of their radii.

75. Two circles will cut each other at right angles if the square of the distance between the centres is equal to the sum of the squares of their radii.

76. To find the radius of a circle, its two parallel chords ($2c$, $2c'$) and the perpendicular distance between them (d) being given.

In fig. 33, $ES = c$, $CR = c'$, $EO = r$, $SR = d$;

let $OR = y$, $OS = x$

Then $x^2 = r^2 - c^2$;

$y^2 = r^2 - c'^2$;

$\therefore x^2 - y^2 = c'^2 - c^2$;

but $x + y = d$ (1)

$\therefore x - y = \frac{c'^2 - c^2}{d}$ (2)

from (1) and (2), $x = \frac{1}{2} \left(d + \frac{c'^2 - c^2}{d} \right)$

$c^2 + \frac{1}{4} \left(d + \frac{c'^2 - c^2}{d} \right)^2$

$c^2 + \frac{1}{4} d^2 + \frac{1}{2} c'^2 - c^2 + \frac{1}{4} \left(\frac{c'^2 - c^2}{d} \right)^2$

Example. Let $CD = 6$, $EF = 8$, and $SR = 7$ feet, find r .

Here, $(SO)^2 = r^2 - 9$

$(OR)^2 = r^2 - 16$

$\therefore SO^2 - OR^2 = 7$

but $SO + OR = 7$

$\therefore SO - OR = 1$

and $SO = 4$, $OR = 3$,

Hence $r = \sqrt{3^2 + 4^2} = \underline{5 \text{ feet.}}$

EXERCISES XIV.

1. In fig. Euc. I. 1, prove that the length of the common chord is $r\sqrt{3}$.

2. In a circle of radius 10, find the height of an arc whose chord is 7.

3. Given the chord 20' and height 4' of an arc of a circle, find the diameter.

4. The height of an arc was found by measurement to be 7 feet $9\frac{1}{2}$ inches; and the chord of half the arc 15 feet 7 inches. With what radius had the arc been described?

5. The span of a bridge, the form of which is an arc of a circle being 48 feet and the height being 10 feet, find the radius.

6. The chord of an arc of a circle is 100 feet, chord of half the arc is 60 feet, find the radius of the circle.

7. The chord of an arc is 8 yards, and the chord of half the arc is 15 feet, find the diameter of the circle. (Cal. Univ.)

8. A man standing on the deck of a ship, 20 feet above the surface of the water, can just see the light of a light-house above the horizon. The light is 120 feet above the level of the sea. Find his distance from the light-house.

(Radius of the earth = 4,000 miles.) (Cal. Univ.)

9. AB and AC are chords of a circle at right angles to one another, their lengths being 30 and 40 feet respectively. Find the height of the arc AC and the diameter of the circle.

(Cal. Univ.)

10. The chord of an arc is 96 ft. and the diameter of the circle is 100 ft.; find the height of half the arc. (Cal. Univ.)

11. Two circles whose radii are 20.5 and 30.4 inches respectively, are successively placed so as to have their centres 10 inches, 51 inches, 9 inches, 50 inches, and 9.9 inches apart. What are their relative positions in the several cases?

12. How far off can the chimney of Roorkee Workshops be seen supposing it to be 110 feet high, and the diameter of the earth to be 7960 miles?

13. Two parallel chords of a circle are 60 and 80 feet respectively, and the distance between them is 10 feet; find the radius.

14. The chord of an arc is 36 ft. and its height 18 ft.; find the diameter.

15. Supposing the earth were spherical and that its diameter is 8000 miles, the distance from Saharanpur to Agra being about 100 miles, find how high vertically a man must ascend at one of these places in order to see the other.

16. Circles whose radii are 16.6 and 13.4 inches are placed so as to have their centres (1) 4, (2) 30, (3) 32, (4) 3, (5) $3\frac{1}{5}$ inches apart. Find the relative positions of the circles in the each case.

17. The chord of an arc of a circle is 10 feet and the chord of half the arc is 6 feet; find the radius of the circle and the height of the arc.

18. The chord of an arc is 45, the chord of half the arc is 25.5. Find the diameter.

19. The common chord of two circles which intersect is 80 ft. and subtends an angle of 60° at the centre of one circle and an angle of 120° at the centre of the other; find the radii and the distance between the centres.

20. The chord of the whole arc is 120 feet, and its height 11 feet; find the chord of half the arc and the diameter.

21. The diameter of a circle is 200 feet long. It is divided into eight equal parts and parallel chords are drawn through the points of division; find the lengths of these chords.

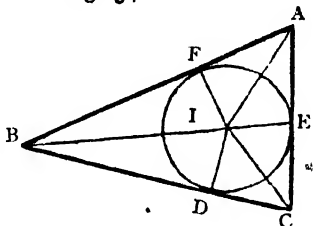
SECTION II.

IN-CIRCLE, EX-CIRCLE, AND CIRCUM-CIRCLE.

Fig. 34.

77. Radius of In-circle.

Let ABC be a triangle, and let DEF be the circle inscribed in the triangle. Let r be radius of the circle; I the centre; D, E, F the points of contact.



$$\text{Then, area of triangle } BIC = \frac{1}{2} BC \times ID = \frac{1}{2} ar$$

$$,, \quad ,, \quad CIA = \frac{1}{2} CA \times IE = \frac{1}{2} br$$

$$,, \quad ,, \quad AIB = \frac{1}{2} AB \times IF = \frac{1}{2} cr$$

$$\begin{aligned} \text{Therefore, adding, area of triangle } ABC &= \frac{1}{2} ar + \frac{1}{2} br + \frac{1}{2} cr \\ &= \frac{1}{2} (a + b + c)r \\ &= sr \end{aligned}$$

$$\text{Hence } r = \frac{\text{Area of } ABC}{s} = \frac{\Delta}{s}$$

[The symbol Δ (read delta) will be used to denote the area of ABC .]

Cor. Substituting for Δ ;

$$r = \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s} = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$$

78. Segments of sides made by the points of contact of In-circle.—

In fig. 34, $BD = BF$, $CD = CE$, and $AE = AF$ and
 $2s = (BD + BF) + (CD + CE) + (AE + AF)$.

$$\therefore BD + CD + AE = \text{half the perimeter} \\ = s$$

$$\text{or } a + AE = s$$

$$\text{Hence } AE = AF = s - a$$

$$\text{Similarly, } BD = BF = s - b$$

$$CD = CE = s - c$$

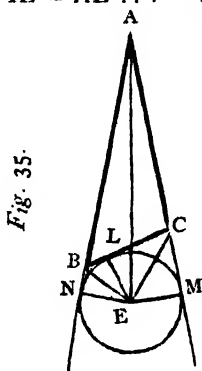
RULE. The segments nearest to A are each $= s - a$ those nearest to B are each $= s - b$, and those nearest to C are each $= s - c$.

When the angle A is a right angle $IE = AE \therefore r = s - c$

79. The ex-circles.

Def. A circle which touches one side of a triangle and the other two produced is called an escribed circle or ex-circle of the triangle.

Let ABC be a triangle, and let LMN be the circle which touches the side BC , and AB , AC produced. Let r_a be the radius of the circle, E the centre; L, M, N , the points of contact.



$$\text{Then, area of triangle } BEC = \frac{1}{2} BC \times EL = \frac{1}{2} a \times r_a$$

$$\text{„ „ } \triangle CEA = \frac{1}{2} CA \times EM = \frac{1}{2} b \times r_a$$

$$\text{„ „ } \triangle AEB = \frac{1}{2} AB \times EN = \frac{1}{2} c \times r_a$$

Therefore, adding the two last and subtracting the first,

$$\text{area of triangle } ABC = \frac{1}{2} br_a + \frac{1}{2} cr_a - \frac{1}{2} ar_a$$

$$= \frac{1}{2} (b + c - a) r_a$$

$$= (s - a) r_a \quad (\text{see art 28})$$

$$\text{Hence } r_a = \frac{\text{area of } \triangle ABC}{s - a} = \frac{s \Delta}{s - a}$$

Similarly r_b (radius of ex-circle touching b) = $\frac{\Delta}{s-b}$

and r_c (radius of ex-circle touching c) = $\frac{\Delta}{s-c}$

80. In fig. 34, $AM = AN$, $BL = BN$, $CL = CM$, therefore

$$AM + AN = AC + CL + LB + BA$$

$$= 2s.$$

$$AM = AN = s$$

$$BL = AN - AB$$

$$= s - c$$

$$CL = AM - AC$$

$$= s - b$$

81. Circum-circle.

Let the circle ABC be described about the triangle ABC and let AD be the diameter of the circle drawn through A.

Draw AE perpendicular from A on BC, and produce BA to F making AF equal to AC, and produce DA to G, making AG equal to AE, join GF and BD.

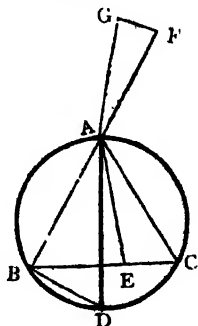
Fig. 36.

Now, in the triangles ABD, AEC, angles ABD and AEC are equal, because each is a right-angle, and the angles ADB and ACE being in the same segment, are equal. Therefore the third angles BAD and EAC are equal. But $BAD = FAG$ (Euc. I. 15) $\therefore FAG = CAE$.

Then, in the triangles FAG, CAE, $GA = AE$, $AF = AC$ and $GAF = CAE \therefore FGA = CEA$ (Euc. I. 4) = one right angle; therefore $FGA = ABD$. Therefore, the points F, G, B, D, are concyclic i.e., a circle will pass through them (Euc. III - 22).

Hence rect. BA. AF = rect. DA. AG

or BA. AC = DA. AE



$$\therefore BA \cdot AC \cdot BC = DA \cdot AE \cdot BC$$

But $AE \cdot BC = 2 \times \text{area of triangle } ABC$

\therefore putting a, b, c for BC, CA, AB respectively, and putting $2R$ for the diameter DA and Δ for the area ABC , we have

$$abc = 2R \times 2\Delta. \quad \text{Hence } R = \frac{abc}{4\Delta}.$$

$$\text{RULE. Circum-radius} = \frac{abc}{4\Delta}$$

$$\text{and Circum-diameter} = \frac{abc}{2\Delta}$$

When the Angle A is a right angle $R = \frac{c}{2}$

Obs. The result comes much more easily by the sixth book of Euclid.

EXERCISES XVI.

1. Find the lengths of the radii of the in-circle, ex-circle and circum-circle in the ten triangles of Exercise VII.
2. Find the diameter of a circle which can be described about a triangle whose sides are 70, 240, and 250 feet respectively.
3. The sides of a triangle are 8, 6, and 10 feet respectively, find (a) its area, (b) the diameter of the circum-circle, (c) the height of the arc cut off by the side 8 feet in length. (Cal. Univ.)
4. The side of an equilateral triangle is 100 feet; find the radius of the circum-circle.
5. The sides of a triangle are 293, 285 and 68; find the diameter of the circumscribing circle.
6. Find the radius of the inscribed circle of a triangle whose sides are 10, 24, and, 26.
7. If a circle be inscribed in the triangle whose sides are 13, 14, 15; find the distance from its centre to the most acute angle.
8. If the altitude of an isosceles triangle is equal to the base, prove that the radius of the circum-circle is five-eighths of the base.

SECTION III.

INSCRIBED AND CIRCUMSCRIBED REGULAR
POLYGONS.

Fig. 37.

82. Inscribed Equilateral Triangle.

Let ABC be an equilateral triangle inscribed in a circle of radius r : to find the length of the side AB.

Through A draw the diameter AD. Join BD. AD bisects BC at right angles (Euc. III. 3). Therefore angle BAD = 30° ,

$$\text{and } AB = \frac{\sqrt{3}}{2} \times AD = \frac{\sqrt{3}}{2} \times 2r = r\sqrt{3}.$$

RULE. Side of an equilateral triangle inscribed in a circle = $\frac{\sqrt{3}}{2} \times$ the radius of circle.

RULE. Radius of circle described about an equilateral triangle = $\frac{1}{\sqrt{3}} \times$ side of triangle.

Also, the area of ABC

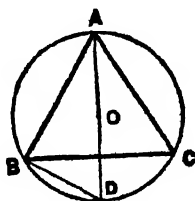
$$= \frac{\sqrt{3}}{4} (AB)^2 \quad (\text{art. 27.})$$

$$= \frac{\sqrt{3}}{4} \times (\sqrt{3}r)^2$$

$$= \frac{3\sqrt{3}}{4} r^2.$$

Obs. The above results can also be deduced from arts. 77 and 81.

83.. Circumscribed equilateral triangle.



Let ABC be an equilateral triangle described about the circle DEF of radius r ; to find AB. Take I Fig. 37a. the centre of the circle. Join ID and IB. Then $\angle IDB$ is a right angle, and $\angle IBD = 30^\circ$

$$\therefore BD = \sqrt{3} \times ID = \sqrt{3} \times r$$

$$\text{Hence } BC = 2 \sqrt{3} \times r$$

$$\text{also } ID = \frac{BC}{2\sqrt{3}}$$

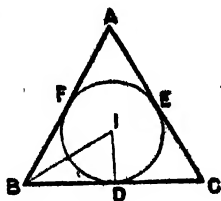


Fig. 37a.

RULE. The side of an equilateral triangle describes about a circle $= 2\sqrt{3} \times$ radius of circle.

Obs. Thus the side of the circumscribed equilateral triangle is double that of the inscribed equilateral triangle.

$$\text{Also, area of ABC} = \frac{\sqrt{3}}{4} \times (BC)^2 = \frac{\sqrt{3}}{4} \times (2r\sqrt{3})^2 = 3r^2\sqrt{3}.$$

Thus the circumscribed equilateral triangle is four times the inscribed equilateral triangle as is evident from figure.

84. Inscribed square.

It is evident from the figure, Euc. IV. 9; that the side of the square inscribed in a circle $= \sqrt{2} \times$ radius of circle.

Hence, radius of the circle described about a square $= \frac{1}{\sqrt{2}} \times$ side of square.

85. Circumscribed square.

It is evident from the figure in Euc. IV. 7, that the side of the square circumscribed about a circle $= 2 \times$ radius of circle.

Hence, radius of the circle inscribed in a square
 $= \frac{1}{2} \times \text{side of square.}$

86. Inscribed regular hexagon

We see from Euc. IV. 15 that the side of a regular hexagon inscribed in a circle is = radius of the circle.

By joining the centre with angular points the hexagon is divided into six equal and equilateral triangles, whence its area $= 6 \times (\text{side})^2 \times \frac{\sqrt{3}}{4} = \frac{3\sqrt{3}}{2} (\text{radius})^2$

87. Circumscribed regular hexagon

Let ABCDEF Fig. 37b. be the regular hexagon circumscribed about the circle LMN, whose centre is O. Join AO, MO.

Now angle BAF $= 120^\circ$ (art. 44)

\therefore angle OAM $= 60^\circ$

and AMO $= 90^\circ$

$\therefore AM = \frac{1}{\sqrt{3}} MO$; but AF $= 2 AM$

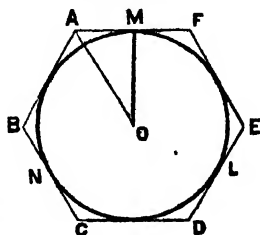
$\therefore AF = \frac{2}{\sqrt{3}} MO = \frac{2r}{\sqrt{3}}$.

RULE. Side of the regular hexagon circumscribed about a circle $= \frac{2}{\sqrt{3}} \times \text{radius of circle.}$

By joining the centre with angular points the hexagon is divided into 6 equal triangles whence its area $= 6 \text{ area of AOF} = 6 \times \frac{1}{2} AF \times OM = 3 \times \frac{2r}{\sqrt{3}} \times r = 2r^2 \sqrt{3}$

88. Def. A Regular Polygon is that which has all its sides equal and also all its angles equal *e.g.* a square, an equilateral triangle.

Fig. 37b.



Def. The *centre* of a regular polygon is the point which is the common centre of the circles inscribed in and circumscribed about the polygon.

Note. This point is equally distant from the sides of the polygon, and also equally distant from the angular points of the polygon. It is the point of concurrence of the bisectors of the angles of the polygon, and also the point of concurrence of straight lines drawn at right angles to the sides from their middle points.

Def. The *radius* of a regular polygon is the radius of the circle circumscribed about the polygon.

Def. The *apothem* of a regular polygon is the radius of the circle inscribed in the polygon. It is equal to the perpendicular from the centre on a side.

EXERCISES.

1. Prove that the angle between two consecutive radii of a regular polygon is equal to the angle between two consecutive apothems.

2. Prove that the angle between two consecutive radii of a regular polygon is supplementary to the angle of the polygon.

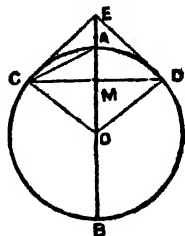
3. The apothem of a square is equal to half the side of the square. Prove this.

89. It will be seen from Euclid's demonstrations in the 12th to 16th propositions of the Fourth Book, that if the circumference of a circle is divided into any number of equal arcs, the inscribed polygon formed by the chords of these arcs is regular; and the circumscribed polygon formed by tangents drawn at the points of division is the circumscribed regular polygon of the same number of sides as the inscribed polygon.

90. Given the side (I) of a regular polygon of n sides inscribed in a given circle, to find the side (I') of a regular polygon of $2n$ sides inscribed in the same circle.

In fig. 38, let $CD (= I)$ be the side of the polygon of n sides and let R be the radius of the circle. Draw the diameter AB perpendicular to CD . Join AC . Then AC is the side of the regular polygon of $2n$ sides inscribed in the circle.

Fig. 38.



Let $AC = I'$.

Then, $AC^2 = AB \cdot AM$

$$= 2R (R - OM)$$

$$= R (2R - 2OM).$$

But $OM = \sqrt{(CO)^2 - (CM)^2}$

$$= \sqrt{R^2 - \frac{I^2}{4}}$$

$$= \frac{1}{2} \sqrt{4R^2 - I^2}$$

Hence $AC^2 = R (2R - \frac{1}{2} \sqrt{4R^2 - I^2})$

and $I' = \sqrt{R (2R - \frac{1}{2} \sqrt{4R^2 - I^2})} \dots\dots\dots (1)$

If R be the unit of length we get from (1)

$$I' = \sqrt{2 - \sqrt{4 - I^2}} \dots\dots\dots (2)$$

And remembering that

$$(2 - \sqrt{4 - I^2})(2 + \sqrt{4 - I^2}) = I^2$$

$$\text{or } 2 - \sqrt{4 - I^2} = \frac{I^2}{2 + \sqrt{4 - I^2}}$$

we get the following form of (2) which is frequently useful in practice.

$$I' = \frac{I}{\sqrt{2 + \sqrt{4 - I^2}}} \dots\dots\dots(3)$$

By repeated applications of the relation (2) or (3) we get successively the sides of regular polygons of $2n$, $4n$, $8n$, $16n$... sides inscribed in the circle of unit radius, n being the number of sides of the original polygon.

We know from art. 84 that the side of a square inscribed in the circle of unit radius $= \sqrt{2} = 1.41421$, whence from the relation (2) the side of a regular octagon inscribed in the same circle

$$\begin{aligned} &= \sqrt{2 - \sqrt{4 - 2}} \\ &= \sqrt{2 - \sqrt{2}} \\ &= .76536. \end{aligned}$$

Hence, starting from the square inscribed in a circle of unit radius we get the following results which are arranged in a table for reference. The student will do well to verify them.

Radius of circumscribing circle = unity.

<i>Number of sides in regular polygon.</i>	<i>Semi-perimeter of reg. polygon.</i>
4	2.82842
8	3.06146
16	3.12144
32	3.13654
64	3.14083
128	3.14127

Again, starting from the regular hexagon inscribed in a circle of unit radius, and remembering that this side is equal to the radius, we get the following series of values, by using formula (2) or (3).

Radius of circumscribing circle = unit.

<i>Number of sides of reg. polygon.</i>	<i>Semi-perimeter of reg. polygon,</i>
6	3.00000
12	3.10582
24	3.13262
48	3.13935
56	3.14103
192	3.14145

91. Given the side (I) of a regular polygon of n sides inscribed in a circle (of radius R), to find the side (C) of a regular polygon of n sides circumscribed about the circle.

In fig. 38, CD = I, and CE and DE being tangents at the extremities of CD, $CE = \frac{1}{2} C$.

Now, OCE being a right angle, and CM perpendicular from C on OE, we have

$$\frac{I}{CE} + \frac{I}{CO} = \frac{I}{CM} \quad (\text{art. 53.}) \dots\dots\dots (I)$$

$$\text{or } \frac{I}{CE} = \frac{I}{CM} - \frac{I}{CO}$$

$$= \frac{I}{\frac{I}{2}} - \frac{I}{R}$$

$$= \frac{4}{I} - \frac{I}{R}$$

$$= \frac{4R - I^2}{I^2 R}$$

$$\therefore CE = \frac{I^2 R}{4R - I^2}$$

and $C = 2CE$

$$= \frac{2IR}{\sqrt{4R^2 - 1^2}} \dots\dots\dots (II)$$

If R be the unit of length

$$C = \frac{2I}{\sqrt{4 - R^2}} \dots\dots\dots (2)$$

Thus knowing the sides of regular polygons of $n, 2n, 4n, 8n\dots$ sides inscribed in a circle of unit radius we can, by formula (2), find the side of circumscribed regular polygons of $n, 2n, 4n, 8n\dots$ sides.

The following table, which the student should verify for himself, gives the sides of circumscribed regular polygons corresponding to the tables in art. 90.

Radius of inscribed circle = unity.

<i>Number of sides of circumscribed reg. polygons</i>	<i>Semi-perimeter of circumscribed reg. polygon.</i>
4	4.00000
8	3.31371
16	3.18260
32	3.15173
64	3.14412
128	3.14223
6	3.46411
12	3.21540
24	3.15967
48	3.14609
96	3.14272
192	3.14188

92. It follows from art. 91 that if I' be the side of an inscribed regular polygon of $2n$ sides, and C' , the side of a circumscribed regular polygon of $2n$ sides

$$C' = \frac{2RI'}{\sqrt{4R^2 - I'^2}} \dots\dots\dots \text{III}$$

93. General Theorems.

If I and C denote respectively a side of the inscribed and circumscribed regular polygon of n sides, and I' and C' a side of the inscribed and circumscribed regular polygon of $2n$ sides, R being the radius of the circle in both cases, the following relations hold:—

$$(1) \quad C = \frac{2RI}{\sqrt{4R^2 - I^2}} \quad (2) \quad I' = \sqrt{2R^2 - R\sqrt{4R^2 - I^2}}$$

$$(3) \quad C' = \frac{2RI'}{\sqrt{4R^2 - I'^2}}$$

$$\text{Also } I = \frac{2RC}{\sqrt{4R^2 + C^2}} \dots\dots\dots (4)$$

$$\text{From (3) } I' = \frac{2RC'}{\sqrt{4R^2 + C'^2}} \dots\dots\dots (5)$$

$$\text{and from (2) } I' = \frac{I'\sqrt{4R^2 - I'^2}}{R} \dots\dots\dots (6)$$

EXERCISES XVII.

1. Find the length of the side of an equilateral triangle inscribed in a circle whose radius is one yard.

2. The radius of a circle is 6 feet: find the side of a square inscribed in the circle.

3. The area of a regular hexagon is 400. Find the length of its side and the radii of its inscribed and circumscribed circles.

-
4. The area of the square circumscribed about a circle is double that of the square inscribed in the same circle. Prove this.
 5. The area of a regular hexagon inscribed in a circle is three-fourths of the circumscribed hexagon. Prove this.
 6. Find the area of a regular hexagon, each side being 30 feet.
 7. The radius of a circle is 5 inches; what is the area of the inscribed regular hexagon?
 8. The radius of a circle is 12 feet; find the length of the side of a regular polygon of sixteen sides inscribed in it. Calculations to be carried to three places of decimals only. (Cal. Univ.)
 9. A square is described about a circle whose radius is $11\frac{1}{2}$ inches: find the area of the square.
 10. A square inscribed within a circle contains $20\frac{1}{2}$ square feet: what is the area of the circumscribed square?
 11. A regular hexagon is inscribed in a circle whose radius is 6 feet: find the area of the hexagon.
 12. The side of an equilateral triangle inscribed in a circle is 30 feet: find the perimeter of the circumscribed equilateral triangle.
-

SECTION IV.

CIRCUMFERENCE OF A CIRCLE.

93. **Rectification of the circle** is the problem of finding a straight line equal in length to the circumference of a given circle.

Since the circumferences of circles are proportional to their diameters, the problem of rectification of the circle is the same as that of finding the ratio of the circumference to the diameter. It is usual to denote this constant ratio by the Greek letter π (from the initial letter of *periphéreia* which is the Greek for *circumference*.) This constant is an *incommensurable* number *i.e.*, it is an interminable decimal. Several mathematicians humorously styled " π computers" have carried the value of π to an astonishing number of decimal places. In 1873, Mr. Shanks computed the value to 707 decimal places.

The following are the first 36 figures of his result :—

3.141,592,653,589,793,238,462,643,383,279,502,884.

"The result is here carried far beyond all the requirements of Mathematics. Ten decimal places are sufficient to give the circumference of the earth to the fraction of an inch, and thirty decimals would give the circumference of the whole visible universe to a quantity imperceptible with the most powerful microscope" *Casey*.

RULE. Circumference = $\pi \times$ Diameter

$$= 2\pi \times \text{radius}$$

$$= 2\pi r,$$

$$\text{or } C = 2\pi R$$

$$\text{and } R = \frac{C}{2\pi}.$$

The student must remember that for all practical purposes it is sufficient to take $\pi = 3.14159$ approximately.

N.B.—As far as three places of decimals $\pi = \frac{22}{7}$, since $\frac{22}{7} = 3.142$.

And, as far as six places of decimals $\pi = \frac{355}{113}$, since $355 \div 113 = 3.1415929$.

Hence, either of the fractions $\frac{22}{7}$, or $\frac{355}{113}$ may be employed instead of 3.14159 according to the degree of approximation required. The latter fraction is easily remembered by the following hint. Write down the first three odd numbers each twice over and divide them in the middle; thus 113/355.

Thus, to find the circumference we must multiply twice the radius by π , and to calculate the radius when the circumference is given, we must multiply half the length of the circumference by $\frac{1}{\pi}$.

Obs. The following value of $\frac{1}{\pi}$ is given for reference only. It need not be learnt by heart.

$$\frac{1}{\pi} = 0.318, 309, 886, 183, 790, 671, 53 \dots$$

Example 1. What is the circumference of a circle whose radius is .65 feet

$$\begin{aligned} C &= 2 \times .65 \times 3.14159 \\ &= 4.08 \text{ feet nearly.} \end{aligned}$$

Example 2. What is the radius of the Meridian of Allahabad? (Given the quadrant of the meridian = 10^7 metres.)

$$\begin{aligned} R &= \frac{C}{2\pi} = 2 \times 10^7 \times \frac{1}{\pi} \\ &= 2 \times 10^7 \times 0.3183099 \\ &= 6366197 \text{ metres nearly.} \end{aligned}$$

94. In this article is attempted an elementary method for finding the value of π after Archimedes who flourished B. C. 287 - 212. The beginner may leave this in his first reading.

It is not difficult to see that the circumference of a circle is greater than the perimeter of an inscribed polygon, and less than that of an circumscribed polygon.

Now, the perimeter of the circumscribed square is four times the diameter (Euc. IV. 8), and the perimeter of the inscribed regular hexagon is 3 times the diameter. Hence π is less than 4, but greater than 3, *i.e.*, it lies between 3 and 4.

By taking $r = \frac{1}{2}$ and proceeding as in art. 90 the side of an inscribed regular polygon of 12 sides = .2588190, and of 24 sides = .1305262 and of 48 sides = .0654031, and of 96 sides = .0327190. Therefore the perimeter of the regular inscribed regular polygon of 96 sides = $96 \times .0327190 = 3.1410240$.

Again from art. 91 we get the perimeter of the regular circumscribed regular polygon of 96 sides to be 3.1427136.

So that π lies between 3.1410 and 3.1427 and therefore as far as the third place of decimals $\pi = 3.141$.

EXERCISES XVIII.

1. Find, correct to four places of decimals the circumference of a circle whose radius is (1) 10 ft. (2) 5 ft. 6 in. (3) 1 mile.

2. How many revolutions are made by a wheel 52 inches high in a journey of 130 miles?

3. The circumference of a circular garden is 286 yards: find the length of a straight path that runs diametrically across the field.

4. The earth describes a circle whose diameter is 187.4 millions of miles in 365 days 6 hours : how many miles an hour does it travel ?

5. The equatorial diameter of the earth is 8,000 miles : how many miles an hour does a place in the equator travel on account of the earth's diurnal revolution ?

6. A horse going round a circus ring 21 feet in diameter makes 15 complete revolutions in a minute : how many miles per hour is the rate of the horse's motion ?

7. How many trees at intervals of $5\frac{1}{2}$ yds. can be planted round a circular park whose diameter is 1260 feet ?

8. The diameter of a circular lawn is 105 yards : find the length of the fence that surrounds the lawn.

9. The cost of fencing a circular garden at Re. 1-2 as. per yard came to Rs. 123-12 as. : find the length of a straight path which bisects the garden.

10. Find the length of the minute hand of a clock, the extremity of which moves over an arc 15 inches in length in $3\frac{1}{4}$ minutes.

11. A man by walking diametrically across a circular grass plot finds that it has taken him 45 seconds less than if he had kept to the path round the outside. If he walks 80 yds. a minute, what is the diameter of the grass plot ? (Cal. Univ.)

12. A square whose side is 50 feet is inscribed in a circle : find the diameter of the circle.

13. The diameter of a circle is 8 feet 4 inches : find the circumference correct to the thousandth of an inch.

14. From a point in the circumference of a circle two chords

are drawn at right angles, and their lengths are 60 and 899 yards : find the circumference of the circle.

15. The circumference of a circular field is 314 yards 10.285714 inches. Find its diameter and area.

N.B.—In this question " π " may be considered $= \frac{22}{7}$.

16. What must be the diameter of a wheel to turn round 800 times in a mile ?

17. Find the length of the minute hand of a clock, the extremity of which moves over an arc 5 inches in length, in $3\frac{1}{4}$ minutes.

18. The extremity of the minute hand of a clock moves 5 inches in 3 minutes. What is its length ?

19. The circumference of a circle is 100 feet : find the length of the side of the inscribed square ; the ratio of the circumference to the diameter is 3.14159 to 1.

[Answer to be correct to two decimal places.] (Cal Univ.)

20. The diameter of a bicycle is $5\frac{1}{4}$ feet ; how far will the bicycle have travelled when the wheel has made a million revolutions ?

21. The circumferences of two concentric circles are 1562 feet and 1386 feet respectively : find the breadth of the circular ring formed by these circles.

22. The radius of a circular plot of ground is 147 feet : what will be the cost of fencing it at Re. 1-4 as. per yard.

23. The minute hand of a clock is $6\frac{1}{2}$ inches long, and the hour hand $4\frac{1}{2}$ inches : compare the rates at which their extremities move.

24. The perimeter of a semicircle is 1125 yards : find the radius of the circle.

25. The horses employed to turn a thrashing-machine move round at the distance of $12\frac{1}{4}$ feet from its centre, and they make 14 circuits in 3 minutes : at what rate per hour do they walk ? ($\pi = \frac{22}{7}$)

SECTION V.

MISCELLANEOUS PROPOSITIONS.

95. The perpendiculars from the angles A, B, C of a triangle ABC on the opposite sides are p, q, r respectively: to find the area.

Let Δ represent the area of the triangle.

$$\text{Then, } \Delta = \frac{1}{2} ap = \frac{1}{2} bq = \frac{1}{2} cr$$

$$\text{and, therefore, } a = \frac{2\Delta}{p}, b = \frac{2\Delta}{q}, c = \frac{2\Delta}{r} \dots\dots\dots (1)$$

$$\text{and } s = \frac{1}{2}(a + b + c) = \Delta \left(\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \right);$$

$$s - a = \frac{1}{2}(-a + b + c) = \Delta \left(-\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \right);$$

$$s - b = \frac{1}{2}(a - b + c) = \Delta \left(\frac{1}{p} - \frac{1}{q} + \frac{1}{r} \right);$$

$$s - c = \frac{1}{2}(a + b - c) = \Delta \left(\frac{1}{p} + \frac{1}{q} - \frac{1}{r} \right);$$

$$\text{But } \Delta = \sqrt{s(s-a)(s-b)(s-c)}$$

$$\therefore \Delta = \sqrt{\Delta^4 \left(\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \right) \left(-\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \right) \left(\frac{1}{p} - \frac{1}{q} + \frac{1}{r} \right) \left(\frac{1}{p} + \frac{1}{q} - \frac{1}{r} \right)}$$

$$\text{or } \frac{1}{\Delta} = \sqrt{\left(\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \right) \left(-\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \right) \left(\frac{1}{p} - \frac{1}{q} + \frac{1}{r} \right) \left(\frac{1}{p} + \frac{1}{q} - \frac{1}{r} \right)}.$$

Thus Δ or the area is known in terms of p, q, r ,

Cor. The sides can also be found from the relations in (1).

Example. The perpendiculars of a triangle are 5 ft., 2 ft., 4 in., 1 ft. 9 in.; find the area and sides of the triangle.

Here $p = 60$ in., $q = 28$ in., $r = 21$ in.

$$\begin{aligned} \therefore \frac{1}{p} + \frac{1}{q} + \frac{1}{r} &= \frac{1}{60} + \frac{1}{28} + \frac{1}{21} = \frac{42}{420}; \\ -\frac{1}{p} + \frac{1}{q} + \frac{1}{r} &= \frac{1}{60} + \frac{1}{28} + \frac{1}{21} = \frac{28}{420}; \\ \frac{1}{p} - \frac{1}{q} + \frac{1}{r} &= \frac{1}{60} - \frac{1}{28} + \frac{1}{21} = \frac{12}{420}; \\ \frac{1}{p} + \frac{1}{q} - \frac{1}{r} &= \frac{1}{60} + \frac{1}{28} - \frac{1}{21} = \frac{2}{420}; \\ \therefore \Delta &= \frac{1}{(420)^2} \sqrt{42 \times 28 \times 12 \times 2} \end{aligned}$$

$$\text{Hence } \Delta = \frac{420 \times 420}{7 \times 6 \times 4} = \underline{1050} \text{ sq. inches};$$

$$\text{and } a = \frac{2 \times 1050}{60} = \underline{35} \text{ inches};$$

$$b = \frac{2 \times 1050}{28} = \underline{75} \text{ inches};$$

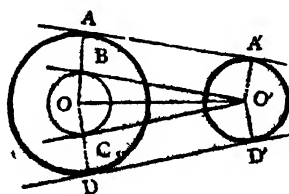
$$c = \frac{2 \times 1050}{21} = \underline{100} \text{ inches}.$$

N.B.—The least perpendicular corresponds to the greatest side.

96. Given the radii of two circles and the distance between the centres to find the length of the **direct common tangents**.

Let AD and A'D' be two circles and let O and O' be their respective centres at a distance a from each other, and let their radii, OA, OA' be equal to r , and r' respectively; suppose r to be greater than r' .

Fig 39.



With centre O and radius equal to $r - r'$ describe the circle BC, and from O' draw O'B, O'C tangents to the circle BC. Join OB, OC and produce them to meet the circle AD in A and D.

Through O' draw O'A' and O'D' parallel to OA, OD respectively. Join AA', DD'. AA', DD' will be the direct common tangents.

For, OA = r , and OB = $r - r'$,

$$\therefore AB = r - (r - r')$$

$$= r'$$

$$= O'A'$$

and O'A' has been drawn parallel to AB, therefore AA' is equal and parallel to OB and the angles at A and A' are right-angles

Hence AA' is a common tangent. Similarly DD' is the other direct common tangent.

Again, AA' = O'B

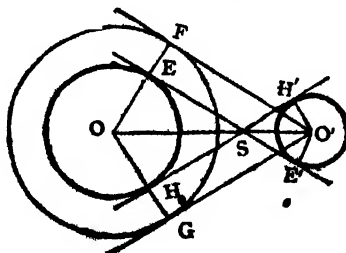
$$= \sqrt{(OO')^2 - (OB)^2}$$

$$= \sqrt{d^2 - (r - r')^2}$$

$$= \sqrt{(d + r - r')(d - r + r')}$$

97. Given the radii of two circles and the distance between the centres : to find the length of the **transverse common tangents**.

Fig. 40.



Let FH , and $E'H'$ be two circles, and let O and O' be their respective centres at a distance d from each other, and let their radii OE , $O'E'$ be equal to r and r' respectively.

With centre O and radius equal to $r + r'$ describe the circle FG , and from O' draw $O'F$, $O'G$ tangents to the circle FG . Join OF , OG , cutting the circle EH in E and H . Through O' draw $O'E'$ and $O'H'$ parallel to OF , OG respectively. Join EE' , HH' .

EE' and HH' will be the **transverse common tangents**.

For $OF = r + r'$, and $OE = r$

$$\therefore EF = r'$$

and $O'E'$ has been drawn parallel to EF .

$\therefore EE'$ is equal and parallel to $O'F$ and the angles at E and F are right angles.

Hence EE' is a transverse common tangent; similarly HH' is the other transverse common tangent. Again

$$EE' = O'F$$

$$\begin{aligned} &= \sqrt{(OO')^2 - (OF)^2} \\ &= \sqrt{d^2 - (r + r')^2} \\ &= \sqrt{(d + r + r')(d - r - r')} \end{aligned}$$

Example. The radii of two circles are 7 and 17 feet respectively and the distance between their centres is 26 feet: find the lengths of their direct and transverse common tangents.

Length of direct common tangent

$$\begin{aligned} &= \sqrt{(26 + 17 - 7)(26 - 17 + 7)} \\ &= \sqrt{36 \times 16} \\ &= 24 \text{ feet.} \end{aligned}$$

Length of transverse common tangent

$$= \sqrt{(26 + 17 + 7)(26 - 17 - 7)}$$

$$= \sqrt{50 \times 2}$$

$$= \underline{10} \text{ feet.}$$

98. Area of a Regular Polygon. Every regular polygon can be circumscribed by a circle. The centre of this circle may be called the centre of the polygon, as the perpendiculars from this point on the sides are all equal, and the distances of the point from the angles of the polygon are all equal. (See art. 88.) It is clear that by joining the centre with the angular points, the polygon can be divided into as many equal triangles as there are sides in the polygon.

Hence.—

RULE. Multiply the perimeter of the polygon by the perpendicular let fall from the centre upon one of the sides, and take half the product for the area.

Example. Find the area of a regular pentagon of which the side is 5 feet, and the perpendicular from the centre on each side is 3.4409547 feet.

Here $5 \times 5 = 25$ is the perimeter.

And $25 \times 3.4409547 = 86.0238675$.

Its half 43.01193375 square feet is the area sought.

Obs. The perpendicular and the side in regular polygons are not independent magnitudes, but the one can always be deduced from the other by Trigonometry.

99. The following table contains the values of the areas of ten regular polygons and the lengths of the radii of their inscribed and circumscribed circles, the length of each side being supposed to be unity.

No of sides in polygon.	Area.	In-radius.	Circum-radius.
3	0.4330127	0.28867	.57735
4	1.0000000	0.50000	.70710
5	1.7204774	0.68819	.85065
6	2.5980762	0.86602	1.00000
7	3.6339124	1.0383	1.1523
8	4.8284271	1.2071	1.3065
9	6.1818242	1.3737	1.4619
10	7.6942088	1.5388	1.6180
11	9.3656407	1.7028	1.7747
12	11.1961524	1.8660	1.9318

Obs. When the side is equal to a , the area is $a^2 \times$ the corresponding number in the column for areas in the above table.

100. **Area of a circle,** From art. 99 it is manifest that if a regular polygon be conceived to be inscribed in a circle the area of the polygon is equal to the product of the semi-perimeter of the polygon and the perpendicular from the centre on any of the sides.

Now, conceive the number of sides of the polygon to be indefinitely increased. Then the perimeter of the polygon becomes the circumference of the circle and the perpendicular from the centre on the side becomes the radius of the circle.

Hence, the area of the circle is equal to the product of the semi-circumference and the radius.

Now, if r be the radius, $2\pi r$ is the circumference (art. 93) and πr is the semi-circumference.

$$\therefore \text{area} = \pi r \times r = \pi r^2.$$

RULE. Area of circle of radius $r = \pi r^2$.

101. **Formulae.** If A denote the area, r the radius, $d = 2r$ the diameter, and C the circumference of a circle, we have.

$$A = \pi r^2 \dots\dots\dots(i)$$

$$= \frac{\pi d^2}{4} \dots\dots\dots(ii)$$

$$= \frac{1}{2} Cr \dots\dots\dots(iii)$$

$$= \frac{C^2}{4\pi} \dots\dots\dots(iv)$$

From (i) $r = \frac{\sqrt{A}}{\sqrt{\pi}} = .5641896 \times \sqrt{A}.$

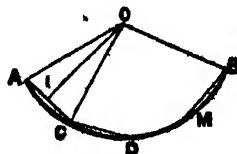
From (ii) $d = \frac{2}{\sqrt{\pi}} \sqrt{A} = 1.1283792 \times \sqrt{A}.$

From (iv) $C = 2 \sqrt{\pi A} = 3.5449077 \times \sqrt{A}.$

102. **Area of a Circular Sector.**

Let $\dot{O}AB$ be a sector of a circle with centre O . Conceive the arc AB to be divided into n equal parts $AC, CD, \dots MB$. Join AC, OC , and draw OI perpendicular to AC . Then the area of the triangle OAC is $\frac{1}{2} AC \times OD$, and the area of the polygon $OACD \dots B = n \times \frac{1}{2} AC \times OD$.

Fig. 41.



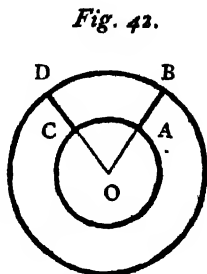
Now, conceive n to be indefinitely increased. Then AC coincides with the circumference of the sector, and $n AC =$ the arc AB , and OI becomes the radius; hence, area of sector $= \frac{1}{2} r \times$ arc of sector.

RULE. Area of sector $= \frac{1}{2}$ radius \times arc.

103. **Area of a Circular Ring.**

Let $ABDC$ be a circular ring, (fig. 42) and r and r' be the radii of the outer and inner circles respectively, then the area of the outer circle is πr^2 and that of the inner $\pi r'^2$, and that of the space between the circles, $\pi r^2 - \pi r'^2$ or $\pi (r + r') (r - r')$.

Obs. In the figure, the circles are concentric, and consequently the ring is of uniform breadth all round since $AB = CD = r - r'$. But the above result holds even when the circles are not concentric.



EXERCISES XIX.

1. The circumference of a circular field is 2000 yds.; find the area of the field in acres, roods and poles.
2. A carriage drive is to be made round the outside of a circular park whose radius is 585 feet. The metalling is to be 30 feet wide and 9 inches deep; what will it cost at Rs. 6 per 100 cubic feet?
3. If a circle has the same perimeter as a triangle the circle has the greater area; verify this statement in the case where the sides of a triangle are 9, 10 and 17 feet.
4. Find the radius of a circle, which, described with one of the angles of a square as centre, divides it into two parts of equal area. The length of a side of the square is 10 feet.
5. A circle whose area is 31416 square inches is to be divided into four equal portions by concentric circle. Find their diameters.

6. Find the area of a segment of a circle, the arc of which is a quadrant, and the diameter of the circle 12 feet.

7. Out of a circular disc of metal 35 equal holes are punched. The weight of metal thus punched out, is to the weight of perforated disc, as 45 : 67 : compare the diameters of the disc and of the holes.

8. The diagonal of a square is 20. Find the area of one of the segments of the circumscribing circle.

9. Find the area in square chains of the circle inscribed in a triangle, of which the sides are 372,350 and 320 yards, respectively.

10. The three sides of a triangle inscribed in a circle are 160, 300 and 340 feet respectively ; find the difference between the area of the circle, and that of the triangle.

11. Prove the following statement:—The area of the space between two concentric circles is equal to the area of a circle which has for its diameter a chord of the outer circle which touches the inner circle.

12. In cutting four equal circles, the largest possible, out of a piece of card-board 10 inches square, how many square inches must necessarily be wasted ?

13. The diameter of a rupee is $1\frac{1}{4}$ inches, if three of these coins be placed on a table so that the rim of each touches two others, it is required to find the area of the unoccupied space between them.

14. Find the areas of a square, a hexagon, and a circle, the perimeter of each being 3,000 feet.

15. From the four corners of a square four quadrants are struck

off each bisecting two sides of the square. Find the area of the "curved square" enclosed by the quadrants, supposing the side of the square to be m feet.

16. The area of a sector is 3,240 and the length of its arc 108. What is the radius?

17. Find the area of a clock-face whose radius is 1 foot 9 inches.

18. Find, in acres and decimals of an acre, the area of the largest circle that can be inscribed in a square whose area is 1,522,756 square feet. Give also the length of its circumference.

19. The side of a regular hexagon is 100 feet. What is the area of the circumscribing circle.

20. A triangle, sides 12, 14 and 18 inches, is inscribed in a circle; find the area of the circle.

21. Find the area of a circular piece of ground, the circumference of which is 20 feet.

22. A circular pond has to be made in a circular enclosure laid out for Botanical Gardens, the greatest breadth of which is 360 yards. The pond is to cost exactly the same sum of money, as the enclosure cost in planting out, &c., but the construction of the pond is twice as expensive per superficial yard. What is the area of the pond in square feet?

23. A circular grass-plot whose diameter is 40 yards, contains a gravel walk one yard wide running round it, one yard from the edge; find what it will cost to turf the grass-plot at 4d. per square yard.

24. The minute hand of a clock is 6 inches long, and the hour hand, 5 inches. How far does the extremity of the former travel

in a year? Also what is the difference of the areas passed over by the hands in the same time?

25. Find the area traversed in 21 minutes by a minute hand which is 9 inches long. What is the area traversed by the hour-hand in the same time, supposing it to be two-thirds as long as the minute-hand?

26. A horse is tethered by a chain 10 feet long fastened to a ring which slides on a rod bent into the form of a rectangle whose sides are 30 ft. and 25 feet. Find the area of the field over which the horse can graze. If the chain be 20 feet long how will the grazing area be altered?

104. EXAMPLES WORKED OUT.

Example 1. If $a = 20$, $b + c = 22$,
and the median $AD = \sqrt{37}$ find b ,
and c .

$$\begin{aligned} \text{By art. 32, } b^2 + c^2 &= 2 \times 37 + 2 \times 10^2 \\ &= 274 \dots\dots\dots (i) \end{aligned}$$

$$\text{and } b + c = 22 \dots\dots\dots (ii)$$

$$\therefore b^2 + c^2 + 2bc = 484$$

$$\therefore \quad \cdot \quad 2bc = 210 \dots\dots\dots (iii)$$

$$\begin{aligned} \text{Subtracting (iii) from (i) } b^2 + c^2 - 2bc &= 64 \\ \text{or } b - c &= 8. \end{aligned}$$

Hence, from (i) and (ii) $b = 15$, $c = 7$.

Example 2. The hypotenuse of a right-angled triangle is 169 feet, and the sum of the other two sides 139; find the value of the radius of the in-circle.

Fig. 43.

In fig. 43 $AB = 169$, $AC + BC = 239$; $IE = ID = IF$, and the angles at E, C, and D are right-angles, $\therefore IE = ID = CD = CE = r$;

But $AF = AE$, and $BF = BD$;

$$\therefore AC + BC = AE + BD + EC + CD = AB + 2r.$$

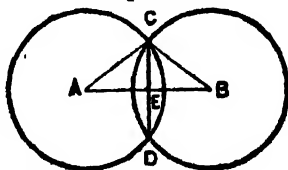
$$= 169 + 2r.$$

But $AC + BC = 239 \therefore 2r = 239 - 169 = 160$ or $r = 80$.

Example 3. The radii of two circles are 25 and 101 feet respectively, and the distance of their centres is 114 ft.; find the length of the common chord.

Fig. 44.

In the triangle ABC, the sides $AB = 114$, $AC = 25$ and $BC = 101$, therefore, by art. 29, the area $ABC = 1140$ sq. ft.



Now, CE is the perpendicular from C on AB,

$$\text{Therefore } CE = \frac{2 \times 1140}{114} = 20 \text{ ft.}$$

$$\text{Hence } CD = 2CE = 40 \text{ ft.}$$

Example 4. The sides of a triangle are $\sqrt{2b^2 + 2c^2 - a^2}$, $\sqrt{2c^2 + 2a^2 - b^2}$, and $3c$: find the area.

In fig. 45 let $BC = a$, $CA = b$, $AB = c$, and let AD, BE, CF be the medians.

Now, by the Theorem of Appollonius, art. 32,
 $AC^2 + AB^2 = 2BD^2 + 2AD^2$

$$\text{or } b^2 + c^2 = 2 \times \left(\frac{a}{2}\right)^2 + 2AD^2$$

$$\therefore 2AD^2 = b^2 + c^2 - \frac{a^2}{2}$$

$$\text{or } 4AD^2 = 2b^2 + 2c^2 - a^2$$

$$\therefore 2AD = \sqrt{2b^2 + 2c^2 - a^2}$$

$$\begin{aligned} \text{But } 2AD &= 2 \times \frac{3}{2} AG \\ &= 3AG. \end{aligned}$$

$$\text{Hence } \sqrt{2b^2 + 2c^2 - a^2} = 3AG.$$

$$\text{Similarly } \sqrt{2c^2 + 2a^2 - b^2} = 3BG.$$

$$\text{Also } 3c = 3AB.$$

Hence the question is the same as finding the area of a triangle whose sides are three times the sides of the triangle ABG.

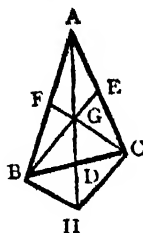
$$\begin{aligned} \text{Therefore the required area} &= 9 \times \text{area of ABG} \\ &= 3 \times \text{area of ABC} \\ &= 3 \sqrt{s(s-a)(s-b)(s-c)}. \end{aligned}$$

Example 5. Prove that the area of a regular dodecagon is three times the square of the radius of the circumscribed circle.

In fig. 38, if CD be the side of a regular hexagon, CA is the side of the regular dodecagon, and area of dodecagon = 12 × area OAC = 6 × area OCAD

$$\begin{aligned} &= 6 \times \frac{1}{2} OA \times CD \text{ (art. 39.)} \\ &= 3 \times OA^2 = \underline{3r^2}. \end{aligned}$$

Fig. 45.



MISCELLANEOUS EXAMPLES.

1. From a point in the circumference of a circle two chords are drawn at right angles, and their lengths are 66 and 77.9 yards respectively: find the radius of the circle.

2. The lengths of two chords of a circle which intersect each other at right angles are 12 and 16 feet respectively, and the distance of their point of intersection from the centre is 5 feet: find the diameter.

3. The chords of two arcs of a circle are 57 and 60 feet respectively and the chord of the sum of these arcs is 111 feet: find the diameter of the circle.

4. The circumference of the earth being 25,000 miles, and the distance from London to York being 200 miles, to what height approximately must a man ascend in a balloon from one of these places in order to see the other?

5. The sides of a triangle are in the ratio 13 : 14 : 15, and the perimeter is 84 yards; find the perpendiculars from the angular points on the sides.

6. The lengths of the two lines drawn from the acute angles of a right-angled triangle to the middle points of the opposite sides are 10 and $4\sqrt{10}$ feet respectively: find the sides of the triangle.

7. A box without a lid, and made of wood 1 inch thick, requires painting inside and out. Its exterior length, breadth, and depth are 3, 2, and $1\frac{1}{2}$ feet respectively. How many superficial feet of paint will be required for each coat?

8. The sides of a triangle are 13, 9 and 6. Find, correct to

three places of decimals, (1) the three altitudes, (2) the three medians, (3) the radius of circum-circle.

9. If a regular hexagon, a square, and an equilateral triangle be inscribed in a circle, show that the square described upon the side of the triangle is equal to the sum of the squares described upon one side of each of the other two figures.

10. The floor of a room, 30 feet long and 24 feet broad, is to be paved with flagstones, all of the same size. If they are all laid down in one direction, the number in the breadth will exceed that in the length by two, while if they are all placed the other way, the number in the length will exceed that in the breadth by seven. Find the number of flagstones required to cover the floor.

11. Prove that the difference of the circumference of a circle and the perimeter of the inscribed hexagon is a little greater than one-fifth of the side of the inscribed square.

12. The area of an equilateral triangle inscribed in a circle is 389.7; what is the area of a square inscribed in the same circle?

13. Find the area of a square inscribed in a quadrant whose radius is $\sqrt{3}$, two sides of the square lying on the bounding radii of the quadrant.

14. Find the side of a regular hexagon, which shall be equal in area to an equilateral triangle, each side of which is 500 feet.

15. The area of an equilateral triangle whose base falls on the diameter and its vertex in the middle of the arc of a semi-circle, is equal to 100. What is the diameter of the semi-circle?

16. Compare the areas of an equilateral triangle, a square, and a regular hexagon of equal perimeter.

17. What is the side of an equilateral triangle which has as many square yards in its area as lineal yards in its periphery?

18. The perimeter of a rectangular field is 434 feet, and the diagonal 175 feet: find the sides.

19. What is the altitude of a parallelogram whose base is 560 feet and area 7,000 square yards?

20. In turning a chaise once round a ring it was observed that the outer wheel made $13\frac{3}{4}$ revolutions while the inner made 11, the wheels being 4 feet $10\frac{1}{2}$ inches asunder; required the diameter of the wheels, and of the circle described by the inner wheel.

21. Give the rules for finding the diameter of a circle with the following data:—

(1). Height of arc, and chord of half the arc.

(2). Sides of an inscribed triangle.

The circumference of a circle is divided into two arcs by a chord measuring 720, the diameter being 900; find the height of the lesser arc, and the chord of half the greater arc.

22. A person standing on the bank of a river observes the angle of elevation of a tree on the opposite bank to be 60° and on receding 80 ft. from the river's bank he observes the elevation to be 30° . Find the height of the tree and the breadth of the river.

23. How many planks of $1\frac{1}{8}$ inches in thickness can be cut out of a piece of timber 21 inches thick, allowing $\frac{1}{8}$ inch for each saw-cut?

24. The two sides of a triangle are 60 and 899 feet. Find the perpendicular on the base if the contained angle is 90° .

25. AC is the diameter of a circle and a diagonal of the inscribed quadrilateral ABCD; given $AB = 30$, $BC = 40$, $CD = 48$: find AD, and the area of the quadrilateral.

26. Find the length of a circular walk which touches the three sides of a triangular field whose sides are 78, 35, and 97 yards respectively. Also find the shortest and longest distances of the walk from the corners of the field.

27. Find the area of a regular octagonal field, each of whose sides measures 5 chains : give the result in acres, roods &c.

(Cal. Univ.)

28. A two-wheeled carriage whose axle-tree is four feet long is driven round a circle ; the outer wheel makes one and a half revolution for every single revolution of the inner one. What is the circumference of the circle described by the outer wheel ?

(Cal. Univ.)

29. Find the side of a regular hexagon equal in area to an equilateral triangle whose side is 900 yards.

30. A carriage wheel, (diameter 28 inches,) makes 360 turns in travelling a certain distance on the level ; how far has any point on the tire of the wheel travelled horizontally ?

31. The hypotenuse of a right-angled triangle is 150, and the perpendicular on it from the right angle 24, calculate the sides.

32. A meteor 50 miles high has an angular elevation of 30° ; find the distance, from the observer, of the place vertically under the meteor.

33. Two adjacent sides of a parallelogram 5 and 8 inches respectively, include an angle of 30° . Find the area and the diagonals.

34. In the triangle ABC, if $a = 2$, $b = 3$, $c = 4$, find the lengths of the medians to six places of decimals.

35. Find the area of the triangle whose sides are

(i) $m^2 + n^2$, $m^2 - n^2$, $2mn$

(ii) $pq(r^2 + s^2)$, $rs(p^2 + q^2)$, $(ps + qr)(pr - qs)$

(iii) $(2a + 1)(a + 1) - 3$, $2(a^2 + 1)$, $5a$.

36. The base of a triangle is 100 yards, and the base angles 60° and 90° ; find the sides.

37. The sides of a triangle are 6105, 6292, 8767; find the area and the radii of the inscribed and circumscribed circles.

38. Find the angles of the triangle in which the sides are as the numbers 1, 2, $\sqrt{3}$.

39. The sides of a triangle are 39, 41, 50: find the area.

40. The base of a triangle is 16, the sum of the other two sides is 20, and the bisector of the base from the vertical angle is 9: find the sides.

41. The sides of a triangle are 50, 78, and 80: find the length of its three perpendiculars.

42. Two equilateral triangles are given whose sides are respectively 24.7 and 159.6: find the side of the equilateral triangle whose area shall be equal to the sum of the areas of the given ones.

43. The sides of a triangle are 2, 17 and 18. Find the area.

44. Find the side of the equilateral triangle whose area is 4,330 square feet.

45. The sides of a triangle are 15, 17 and 19: find the area.

46. The sides of a triangle are 119, 120, 169: find the ratio of the segments into which the side 169 is divided by the perpendicular dropped from the opposite angle.

47. In a triangle ABC, $a = 78$, $b = 82$, $c \approx 100$: find the area.

48. The sides of a triangle are 114, 101, and 25: find the area.

49. If the sides of a triangle be 84, 96 and 108 feet respectively, calculate the segments into which the greatest side is divided by the perpendicular from the opposite angle.

50. The sides of a triangle are 3:3, 4:4, and 5:5: find the length of the perpendicular from the intersection of the first two on the last.

51. In a triangle ABC, $a = 7$, $b = 8$ and $c = 9$: find the medians.

52. The sides of a triangle are 125, 123 and 62: find the area.

53. The sides of a triangle are 187, 200 and 123: find the length of the least perpendicular.

54. A regular hexagon with hinges at its angular points is bent into the shape of an equilateral triangle, find the ratio of the areas of the two figures. Find also the ratio of the areas of their inscribed circles.

55. The sides of an isosceles triangle are 15, 15, 26. The base 26 is divided at a point P into segments 5 and 21. Find the distance of P from the vertex.

56. Three circles whose radii are 6, 7, and 8 feet respectively touch each other. Find the area of the triangle formed by joining their centres.

57. The sides of a triangle are 287, 1,000, and 1,023: find the radius of the in-circle.

58. Find the length of the radius of the circle which is circumscribed about an equilateral triangle whose area is 100 square feet.

59. An observer standing on a mountain 700 feet high just barely sees above the horizon across the sea the top of another mountain 1,400 feet high. Find the distance of the mountains, proving the formula employed.

60. The perpendiculars of a triangle are 2,050, 1,599 and 1,950: prove that the sides are proportional to the numbers 39, 50, and 41 respectively and find the area.

61. A triangle is given whose sides are 19, 27, 34 : find the distance of the centre of the inscribed circle from the angles.

62. Find the radii of the inscribed and circumscribed circles of the triangle whose sides are 1.13, 2.38, and 2.25 feet.

63. $a = 42.12$, $b = 90.09$, $c = 99.45$ find R.

64. $a = 91$, $b = 312$, $c = 325$. Find R and the perpendicular let fall on 325 from the opposite angle.

65. The sum of the perimeters of two circles is to their difference in the ratio of 25 : 16 and the difference of their areas is equal to the area of a circle whose radius is 4 feet : find the radii of the circles.

66. If Δ be the area of a given triangle and Δ' the area of another triangle whose perpendiculars are the reciprocals of the sides of the first triangle, prove that $\Delta\Delta' = \frac{1}{4}$. (See art. 95.)

67. Show how to cut off the corners of a square so that the figure which remains shall be a regular octagon.

68. If $a = 200$, $b = 187$, $c = 123$: find R.

69. The top of a mountain 6,600 feet high can just be seen at sea 100 miles off : required the diameter of the earth.

70. At the distance of twenty miles from a tower its top just appeared in the horizon : determine its height.

71. Find the length of the diameter of a circle whose circumference is 246.36148739.

72. The sides of a triangle are 5, 8, 12 : find the space included between the inscribed and circumscribed circles.

73. The perpendiculars of a triangle are 3, 4 and 6 : find the sides.

74. If $a = 51$, $b = 75$, $c = 78$ find R .

75. The sides of a triangle are 10 yds. 26 ft. and 28 ft. respectively: find the value of the radius of the circum-circle.

76. The radii of two circles are 1 and 7 and the distance between their centres is 10. Calculate the lengths of the common tangents.

77. The radius of a circle is 10 feet; from a point P at a distance of 17 feet from the centre, a line is drawn cutting the circle in A and B ; find the area of the rectangle under PA , and PB .

78. If the angle in a segment of a circle be 135° , find the ratio of the arc of the segment to the whole circumference.

79. The hypotenuse of a right-angled triangle is 12.5 and the sum of the sides 17.5: calculate the value of r .

80. The radii of two circles are 4.25 and 1.75 feet respectively and the distance between their centres 6.5 feet: find the lengths of their direct and their transverse common tangents.

81. Two parallel chords are 12 metres and 16 metres respectively; and their distance apart is 2 metres: find the length of the diameter.

82. Find how far a point must be from the centre of a circle (radius 10 inches) so that the rectangle under the segments of a chord passing through it may be to the square of the radius in the ratio of 576 : 625.

83. The radii of two circles are 10 feet and 8 yards respectively and the distance between their centres is 26 feet: find the length of their common chord.

84. If $a = 25$, $b = 39$, $c = 56$: find area, r and R .

85. A uniform path running round a circular field contains as much as $\frac{9}{16}$ of the field. Compare the breadth of the path with the radius of the field.

86. Find the length of the shortest chord through a point 21 feet from the centre of the circle whose diameter is 50 yards.

87. What is the magnitude of the rectangle of the segments of a chord drawn through a point 73 feet distant from the centre of a circle whose diameter is 170 feet?

88. Being given a circle whose radius is 10 feet: find the rectangle under the segments of a chord which passes through a point at a distance of 7.33 feet from the centre.

89. Find the angle in a segment whose arc is a quadrant.

90. In the triangle ABC, $a = 26$, $b = 25$, $c = 17$: find r .

91. If a point be h feet outside the circumference of a circle whose diameter is 7.920 miles, prove that the length of the tangent drawn from it to the circumference is $\frac{\sqrt{3h}}{\sqrt{2}}$ miles.

92. The perimeter of a right-angled triangle is 60 and the radius of the inscribed circle, 5: find the sides.

93. The sides of a triangle are 25, 39 and 40. Calculate the radii of the inscribed and circumscribed circles.

94. Find the area of the regular dodecagon inscribed in the circle whose radius is 30.5 inches.

95. The diameter of a circle is $10\sqrt{3}$: find the side of the equilateral triangle inscribed in it.

96. If $a = 100$, $b = 82$, $c = 78$: find r .

97. The length of two parallel chords are $2a$ and $2b$ and the distance between them is c : find r .

98. $a = 25.2$, $b = 53.9$, $c = 59.5$, find R .

99. The three sides of a triangle are 23.4, 25.2, and 27: calculate R .

100. The three perpendiculars of a triangle are 18.2, 19.5 and 21: calculate the area.

101. Find the diameter of the circle circumscribed about the triangle whose sides are 85, 204, 221.

102. The radii of two circles are 84.5 and 97.5 and the distance between their centres is 91: calculate the length of the common chord.

103. The sides of a triangle are 143, 132, 55: find r .

104. If the arc of a segment be one-sixth of the whole circumference, find the magnitude of the angle in the segment.

105. The sides of a triangle are 112, 441, 455: find the perpendicular dropped on the side 441 from the angle opposite to it.

106. The sides of a triangle are 4, 13, and 15 feet: find the lengths of the perpendiculars from the circum-centre on the sides.

107. If $a = 21$, $b = 89$, $c = 100$: find r and the distance of the in-centre from the angular points.

108. Two sides of a triangle are 50 and 24 feet and the contained angle is equal to half the angle of a regular hexagon: find the area.

109. The sides of a right-angled triangle are 3 and 4: what are the segments into which the hypotenuse is divided by the perpendicular from the opposite vertex?

110. The sides of a right-angled triangle are 85 and 204: find the hypotenuse and the perpendicular from the right angle let fall on it.

111. Two sides of a triangle are represented by 1707.37424

and 1067.1089 and the angle between them is half the angle of an equilateral triangle: find the area.

112. The sides containing the right angle of a right-angled triangle are 56.88 and 23.7: find the perpendicular from the right angle on the hypotenuse and the segments of the hypotenuse made by the perpendicular.

113. The sides of a triangular field are 15 perches $4\frac{1}{2}$ yards, 18 perches 1 yard, and 26 perches. What is the length of the side of the inscribed square tank, one side of which lies upon the greatest side of the triangle?

114. Prove that 3, 4, 5 form the only triad of consecutive numbers that can represent the sides of a right-angled triangle.

115. Determine all the ways in which the space round a point can be completely filled by the angles of regular polygons.

116. Find the area of a regular polygon of 24 sides inscribed in a circle, the radius of which is 100 feet.

117. Find the area of the largest circular plate that can be cut out of a square sheet of iron containing 25,281 square inches.

118. A room is 20 feet long, 12 feet broad, and 9 feet high: what length of string will reach from any corner of the floor to the furthest corner of the ceiling?

119. Find the circumference of a circle having an arc whose chord is $1\frac{1}{2}$ feet and height 3 feet.

120. The sides of a triangle are in the proportion of 13, 14 and 15, and the perimeter is 70 yards: find the area.

121. The height of the peak of Teneriffe is 12,000 feet. How far is it visible from a ship at sea. (radius of earth = 4,000 miles)?

122. The sides of an isosceles triangle are each 10 feet and the base angle is 30° . Find the base and the altitude.

123. Two circles of radii 200 and 210 feet respectively intersect, and the distance between their centres is 290 feet: find the length of the common chord and the area of the quadrilateral formed by joining the centres to the points of intersection of the circles.

124. A string passes round two equal circles which touch externally. Find the area enclosed by the string. (radius = 10 ft.)

125. A ladder 25 feet long is placed against a wall with its foot 7 feet from the wall: how far should the foot be drawn out so that the top of the ladder may come down by half the distance that the foot is drawn out?

126. The fence of an octagonal inclosure cost Rs. 75 at 2 as, 6. p. per foot. What will be the cost of turfing the surface at 2 as., per square yard?

127. The side of a square is 16 feet, and its corners are cut off so as to form a regular octagon: find the area of the octagon.

128. An equilateral triangle and a square have the same area: compare their perimeters.

129. A rectangle is 4 inches longer than a certain square, and 2 inches narrower, but contains the same area: find the side of the square.

130. In cutting a circle, the largest possible, out of a cardboard 18 inches square, how much must necessarily be wasted?

131. Find the number of degrees in each of the angles of the isosceles triangle constructed in Euc. IV. 10.

132. A pleasure-garden in the form of a square whose area is

20 acres has in its centre a circular tank, whose circuit is 880 yards: find the length of the paths reaching from each of the corners to the edge of the tank.

133. Find the area of a quadrilateral, the diagonals being 50 and 75 feet, and being at right angles to one another.

134. The wheels of a carriage each 4 feet high, in turning round a ring, moved so that the outer wheel made three turns while the inner made two and their distance from one another was 5 feet. What were the lengths of the tracks described by them?

135. The sides of three squares are 6, 8, and 24 feet respectively; find the side of a square equal in area to the sum of the areas of all the three squares. Shew how to do this mechanically with cardboard and scissors.

136. Give the rules for finding the diameter of a circle with the following data:—

(1) Height of arc, and chord of the arc.

(2) Lengths of three tangents intercepted between their points of intersection.

The circumference of a circle is divided into two arcs by a chord measuring 720, the diameter being 1,681; find the height of the lesser arc, and the chord of half the greater arc.

137. The two parallel chords of a circular zone are 6 and 8 ft. and the diameter of the circle 5 feet: find the breadth of the zone. (This question admits of two answers).

138. An equilateral triangle and a square have the same perimeter: compare their areas.

139. Two circles whose radii are 29.7 and 30.4 inches respectively cut at right angles: find the distance between their centres.

140. Find the area of a trapezium whose parallel sides are 72, and $38\frac{1}{2}$ feet, the other sides being 20 and $26\frac{1}{2}$ feet.

141. The radii of two circles are 36 and 61 feet respectively, and the distance between the centres is 65 ft. : find the length of the common chord.

142. The hypotenuse of a right-angled triangle is 50 feet and the radius of the inscribed circle is 10 feet : find the sides.

143. The radius of a circle is 50 feet : find the areas of the inscribed and circumscribed regular hexagons.

144. The radius of a circle is one foot : find the area of a regular dodecagon inscribed in the circle.

145. The perimeter of a right-angled triangle is 450 feet and the radius of the in-circle is 30 feet : find the sides.

146. A square and a regular hexagon have the same perimeter : compare these areas.

147. The width of a circular walk is not known, but the length of the line which is a chord of the outer circumference of the walk and a tangent to the inner circumference is 20 feet. Find the area of the walk.

148. The side of a square is 12 feet ; the square is divided into three equal parts by straight lines parallel to a diagonal ; find the perpendicular distance between the parallel straight lines.

149. Find the cost of laying out a walk between two concentric fences, 25 and 19 yards in length, at 6 as., per square yard.

150. The diagonals of a quadrilateral are 50 and 60 yards and they intersect at an angle of 30° ; find the area.

151. The shorter diagonal of a regular hexagon is $2\sqrt{3}$ feet ; find the other diagonal, the side, and the area of the hexagon.

152. The side of a square is 100 feet, and its corners are cut off so as to form a regular octagon : find the area of the octagon. Prove that the side of the octagon $= (\sqrt{2} - 1) \times$ side of the square.

153. The area of a regular octagon is 1086.4 feet ; find the length of one side. *

154. An oblong room is 21 feet 7 inches long, 15 feet wide, and 10 feet high. In it are two doors $7' \times 3'$, two windows $5' \times 3'$ with semi-circular heads, and a fire place $4' \times 3' 6''$. What is the surface of the walls ?

155. The radius of a circle is 45 feet : find the areas of the inscribed and circumscribed equilateral triangles.

156. Prove that the area of the inscribed dodecagon is three-fourths of that the circumscribed square.

157. The sides of a triangle are 13, 14, 15 : find the lengths of its medians ; also the lengths of its perpendiculars, and prove that all its angles are acute.

158. A length of fencing of 1000 yards is allotted to enclose a cricket ground. What is the largest area that can be enclosed ?

159. A regular dodecagon is inscribed in a circle of which the radius is 3 inches, find the area of the polygon in square feet.

160. The sides of a triangle are 17 and 19, and the contained angle is equal to half of the angle of an equilateral triangle : find the area.

161. If, in a right-angled triangle m = the bisector of the hypotenuse, d = bisector of the right-angle, and p the perpendicular on the hypotenuse, prove that $\frac{1}{m} + \frac{1}{p} = \frac{2p}{d^2}$.

162. If three given numbers represent the sides of a right-angled triangle, the numbers obtained from them by moving the decimal point the same number of places in each shall represent the sides of a right-angled triangle.

163. If the base AB of a triangle be divided in O into $m + n$ equal parts of which AO contains n and BO, m parts, prove that if the vertex C be joined to O, $m \cdot AC^2 + n \cdot BC^2 = (m + n) \times OC^2 + m \cdot AO^2 + n \cdot BO^2$.

164. A rectangular area is one acre in extent and its perimeter is 322 yards : find the length of its sides.

165. In a triangle ABC, $AB = 100$, $BC = 17$ and the angle ABC is equal to half the angle of an equilateral triangle : find the area.

166. The sides of a triangle being .9, 1.2, and 1.8: find the segments made by the perpendicular on the side 1.8.

167. The base of a triangle is 10 feet long and the bisectors of adjacent sides respectively 8 feet and 9 feet: find the two sides and the other bisector, each to one decimal place.

168. The sides about the right angle of a right-angled triangle are $\frac{1}{36}$ and $\frac{1}{77}$: find the perpendicular from the right angle on the hypotenuse.

169. If the hypotenuse of a right-angled triangle be .35 and one of the sides .2: find the perpendicular from the right angle on the base to three places of decimals.

170. The sides of a triangle are 14.20 and 24. Find the bisectors of the sides to two decimal places.

171. AC is the diameter of a circle and a diagonal of the inscribed quadrilateral ABCD; given $AB = 30$, $BC = 40$, $CD = 14$: find AD and the area of the quadrilateral.

172. The perimeter of a right-angled triangle is 90 and the radius of the inscribed circle is 4. Find each side.

173. A drawing-room is in the form of a regular hexagon, whose side is 16 feet and height 20 feet: find the cost of matting the floor at Rs. 4 per 50 square feet and of painting the walls at 5 as. per square foot.

174. The perpendicular let fall from the right angle on the hypotenuse of a right-angled triangle is 810.948923076 and one side is 1184: find the other sides.

175. A quadrilateral field has four sides equal to 115, 169, 391, and 409 perches respectively and the diagonal which forms the common base to the first two and the last two sides is 120 perches: calculate its contents in acres, roods, and perches.

176. Find, to two decimal places, the length of the three lines joining each vertex to the middle point of the opposite side of a triangle whose sides are 22, 30 and 36.

177. The sides of a right-angled triangle being 3 and 4 feet respectively, calculate to three places of decimals the length of the four lines connecting the vertex of the right angle with the opposite angles of the two squares that may be described on the hypotenuse.

178. Find the area of a regular octagon inscribed in a square the side of which is $10\sqrt{2}$.

179. A Gunter's chain is too long by half an inch. What is the real area of a field which is found to be $68\frac{1}{2}$ acres when measured by this defective chain?

180. If the base of a triangle be 5 feet and its area 20 square feet; find the length of the side of its corresponding inscribed square.

181. A square field is bounded by a path $\frac{1}{2}$ yards wide, the field and path together occupying 50176 square yards: find the cost of gravelling the path at 3 as., per sq. yd?

182. The perpendiculars of a triangle are 6, 7.2, and 9 feet respectively: find the area and the sides of the triangle.

183. Prove that in the five triangles the measure of whose sides are (1) 6, 8, 10 (2) 5, 12, 13, (3) 6, 25, 29, (4) 7, 15, 20, (5) 9, 10, 17, the perimeter and area in each case are denoted by the same figures.

184. A field is 240 cubits long and 180 cubits broad: shew that a road 60 cubits broad running half round inside the field will cover an area equal to half that of the field.

185. Inscribe the least square in a given square.

Find the side of the least square inscribed in the square whose side is 10 feet.

(apply Euc. II. 9)

186. Circles whose radii are 25.2 and 27.5 inches are successively placed so as to have their centres 1.5, 2.3, 25.5, 37.3, 45.7, 52.7, and 54.2 inches apart. What are the relative positions of the circles in each case?

187. A point is 14.7 inches from the centre of a circle of 10 inches radius: find the length of the shortest line from the point to the circle.

188. The diameter of a semicircle is 52, and the length of the

perpendicular, from a point on the curve, on the diameter is $4\sqrt{30}$; find the two parts into which the diameter is divided.

189. Find the radius of a circle which passes through the centres and points of intersection of two circles whose radii are 90 and 400 feet respectively.

190. What is the shape, and what is the area of the largest plane figure that can be enclosed by a mile of rope?

PART THIRD.

[Surveying and Field-Book.

SECTION I.

SURVEYING.

105. **Land Surveying** is the art of determining the area of a given portion of the earth's surface and of delineating the true boundaries thereof in a Plan or Map.

106. There are two methods, by which the dimensions of any piece of land can be taken, viz:—1st. By the chain only; 2nd. By the chain accompanied with angular measurements.

In this little book we shall confine ourselves to the Chain Survey only

107. There are two kinds of chain used for surveying. The one is 100 feet long. The other, usually called Gunter's chain, is 66 feet long. Both chains are divided into 100 links. Gunter's chain is the measure adopted in the Indian Revenue Surveys. The great advantage that Gunter's chain possesses over the other is the facility with which the areas of any field measured by it can be calculated. (See p. 6).

Thus, 1sq. chain = 484sq. yds. = 10,000 sq. links; and 1 acre = 10 sq. chains = 4,840 sq. yds. = 100,000 sq. links.

108. **To measure a straight line with the chain.** Marks (*e. g.* flags, &c) are first set up at the extremities of the line to be measured. Two men hold the chain, one at each end; the foremost or leader is provided with ten arrows, pointed at one end, and having a ring at the other. The leader starts with the ten arrows in his left-hand and one end of the chain in his right. The follower holding the other end of the chain, at the starting point, puts the leader in a straight line with the mark at the other extre

imity of the line to be measured. The leader then puts one arrow in the ground and proceeds a second chain's length in the same direction, while the follower comes to the arrow first put down. A second arrow being now put down by the leader, the first is taken up by the follower. The same process is repeated till the leader has put all his arrows into the ground. Ten chains having now been measured and noted in the field book, the follower returns the ten arrows to the leader and the same operation is repeated as often as necessary until the second station is arrived at. The number of arrows in the follower's hand shews the number of chains measured since the last exchange of arrows. This number together with the odd links between the last arrow and the second station and the number of exchanges for ten chains noted in the field book will make up the entire length of the line.

Example :—Suppose the arrows have been exchanged five times, and the follower has six arrows in his hand and the extremity of the line is at the 75th link from the last arrow put into the ground ; what is whole length of the line ?

Total length = $5,000 + 600 + 75 = 5,675$ links.

109. Offsets. In surveying with the chain, the positions of objects on both sides of the chain line are ascertained by their perpendicular distances from the chain line. These distances are called *offsets*.

Thus, in figure 19, page 56, if $a C b c D f$ be the chain line, Aa , Bb , Ee , and Ff , drawn perpendiculars on the chain line from the points A , B , E , F , are called the offsets of those points.

As has been explained in arts 14, 26, 37 and 41, the areas of the portions between the chain line and successive offsets can be found by the rules for the triangle and trapezium.

110. Offsets are generally measured by a tape; but if they are very short, they may be measured by a graduated offset-staff about 10 feet long.

111. In taking an offset with the tape from an external point, advantage is taken of the following useful geometrical theorem, viz:—"The shortest distance between a straight line and a point lying externally is the perpendicular from the point on the line."

One end of the tape is fixed at the external point, and a sufficient length of the tape is taken and the other end passed along the chain until a point in the chain is obtained which requires the least length of the tape to join the external point. This affords a ready and expeditious method, practically employed by surveyors for taking offsets. The cross or optical square is also employed in measuring offsets.

112. To erect a perpendicular on the chain line from a point in it. This operation can be done with the help of the chain only.

Let BC be the chain line, and C a point in it. It is required to erect a perpendicular on BC at C with the help of the chain only. Fix the end of the chain at C with an arrow at C; fix also the 8th link reckoning from C, at B which is 4 links from C. Take hold of the extremity of the 3rd link from C and pull out the chain so as to make it assume the position CAB, the parts AC, BC remaining tight. CA is then the perpendicular from C on BC (Eu. I 47).

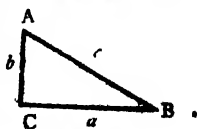


Fig. 46.

It is easy to see that the length 30, 40 and 50 may be employed instead of 3, 4, and 5. The student will have no difficulty in suggesting other numbers.

SECTION II.

THE FIELD-BOOK.

113. The book in which all the steps of the operation of a chain survey are entered is called the **field-book**.

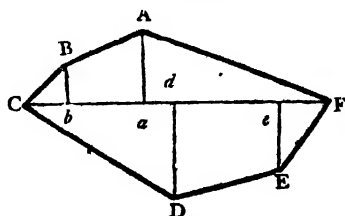
Each page of the field-book is divided into three columns. In the middle column are set down the distances on the chain line, at which offsets stations or other marks are made; and in the right and left hand columns, those offsets &c. are entered according as they are on the right or left chain line. The entries are commenced at the bottom of the last page of the field-book and are written upwards.

114. In the following example the field A B C D E F is surveyed by measuring one chain line from C to F.

Example. Draw a plan of a field, and find its area from the following notes.

Fig. 47.

The points	□ F	
C and F are	250	
called stations	200	45 E
and are marked	120	75 D
□ in the	100	
field book.	50	
From	25	30
In the middle	□ C	



In the middle column the distances are all from the stations C and *not* intermediate distances.

We find from the notes that $Cb = 30$, $Ca = 100$, or $ab = 70$; $Cd = 120$; $Ce = 200$, or $de = 80$; $CF = 250$ or $aF = 50$; $Bb = 25$; $Aa = 50$, $Dd = 75$; $Ee = 45$; $aF = 150$:

Now area A B C D E F = $CbB + BbaA + AaF + CdD + DdeE + EeF$.

$$\begin{aligned}
 &= \frac{1}{2} \left\{ \begin{array}{l} Cb \times bB + ab \times (Bb + Aa) + Aa \times aF \\ + Cd \times dD + de \times (Dd + Ee) + Ee \times eF \end{array} \right\} \\
 &= \frac{1}{2} \left\{ \begin{array}{l} 30 \times 25 + 70(25 + 50) + 50 \times 150 + 120 \times 75 \\ + 80(75 + 45) + 45 \times 50 \end{array} \right\} \\
 &= \frac{1}{2} \left\{ \begin{array}{l} 750 + 5,250 + 7,500 + 9,000 + 9,600 + 2,250 \end{array} \right\} \\
 &= \frac{1}{2} \times 34,350 = \underline{17,175} \text{ sq. links.}
 \end{aligned}$$

Obs.—The student will find it more convenient to calculate double areas of the triangles and trapezium and then halve the whole result.

115. In art. 114 the field A B C D E F has been shown to have been conveniently surveyed by measuring a straight line from one corner to another and taking offsets to this line. Such fields are, however, not common. In the generality of cases, the whole area to be surveyed is divided into a series of triangles, the sides of these triangles being carefully measured with the chain and offsets being taken to all the corners and bends of the field and to objects whose positions are to be shown on the plan. It will be seen that in all cases the entries and calculations are just like those exemplified in art. 114.

116. **The line or test line.**—A line measured to test the accuracy of a survey is called a tie line or test line.

After the triangulation, or division into triangles, of the field, the sides of each triangle are first measured, and as a necessary check a straight line from a definite point on one side to a definite point

on another side is measured (*e. g.* from one of the vertices to a point in or near the middle of the opposite side). This line is an efficient means of detecting errors if any have been committed in the measurement of the sides of the triangle. The tie line affords an independent means of ascertaining the accuracy of the survey.

117.

Example.—Required the plan and area of a field from the accompanying field-notes.

D	60	°	□ E		° C.	40 B	°
			450				
			300				
			200				
			160				
			000				
			□ A				
<hr/>							
N	60	°	□ A		°	30 Q	40 P
			481				
			415				
			360				
			320				
			240				
L	30	°	180				
			150				
K	50	°	000				
			□ H				
Turn	.		to the		left		
<hr/>							
			□ H		°	86 G	70 F
			589				
			450				
			120				
			000				
			□ E		°		

Fig 48.

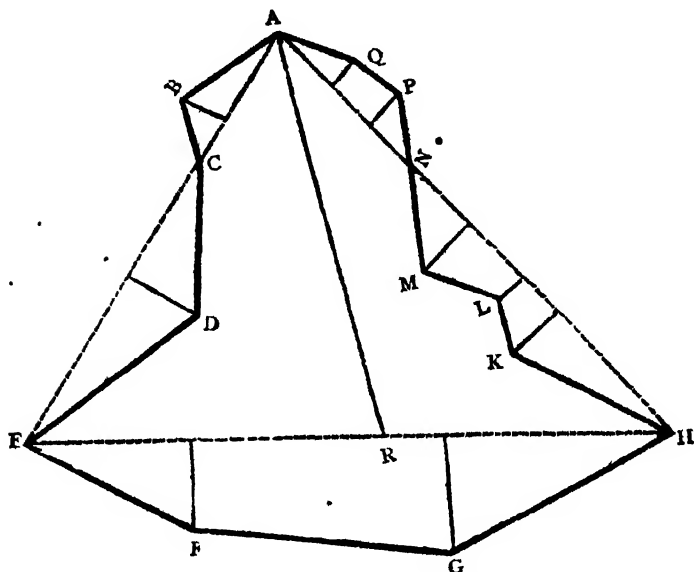


Fig. 48 shows the rough plan of a field ABCDEFGHJKLMNPQ corresponding to the field notes.

The sides of the triangle AEH are 450, 589, and 481, whence its area is 106020 (art. 30).

$$\begin{aligned} \text{As in art 114, } \frac{1}{2} \left\{ 120 \times 70 + 330 \times (70 + 86) + 139 \times 86 \right\} \\ \text{area EFGH} &= \frac{1}{2} \left\{ 8,400 + 51,480 + 11,954 \right\} \\ &= 35,917 \end{aligned}$$

$$\begin{aligned} \text{Similarly, area } \frac{1}{2} \left\{ 150 \times 50 + 30(30 + 50) + 60(30 \times 60) \right\} \\ \text{HKL MN} &= \frac{1}{2} \left\{ 7,500 + 2,400 + 10,800 \right\} \end{aligned}$$

$$= \frac{1}{2} \left\{ 7,500 + 2,400 + 5,400 + 4,800 \right\}$$

$$= 10,050.$$

$$\text{and, area NPQA} = \frac{1}{2} \left\{ 40 \times 40 + 55(30 + 40) + 66 \times 30 \right\}$$

$$= \frac{1}{2} \left\{ 1,600 \times 3,850 + 1,980 \right\}$$

$$= 3715.$$

$$\text{and area ABC} = \frac{1}{2} \times 200 \times 40$$

$$= 4,000$$

$$\text{and area CDE} = \frac{1}{2} \times 250 \times 60$$

$$= 7,500$$

Now area ABCDEFGHKLMPQ

$$= \text{area AEH} + \text{area EFGH} + \text{area NPQA} + \text{area ABC} \\ - \text{area HKLMN} - \text{area CDE}$$

$$= 106,020 + 35,917 + 3,715 + 4,000 - 10,050 - 7,500$$

$$= 132,102.$$

If the measurement be in links

area = 1 ac. 1 ro. 11 perchs. nearly.

Question. The tie-line AR measured from A to R (where ER = 336) measures 366. Does it prove that there has been no error in measuring the sides of the triangle AEH?

Yes. For by Apollonius' Theorem, art. 32,

$$253. AE^2 + 336. AH^2 = 253. ER^2 + 336. HR^2 + 589. AR^2$$

$$\text{whence } 589. AR^2 = 253 (450^2 - 336^2) + 336 (481^2 - 253^2)$$

$$= 253 \times 786 \times 114 + 336 \times 734 \times 228$$

$$= 36 \times 19 \times 115 \ 351;$$

$$\text{or } AR^2 = 36 \times 3721$$

$$\text{Hence } AR = \underline{366}$$

EXERCISES XX.

I. Draw plans and find the areas of fields from the following field notes. (The lengths are all in links.)

(1)	□E	
	360	
C40	200	90B
D100	80	
	□A	

(2)	□C	
	400	
	310	25D
	250	60E
B40	100	
	□A	

(3)	□D	
	450	
C80	340	
	210	60E
B70	120	
	□A	

(4)	□C	
	330	
	250	95D
B70	130	
	75	45E
	□A	

(5)	□E	
	500	
D160	440	
C110	320	
	240	50F
B60	100	
	□A	

(6)	□E	
	1024	
D96	973	
C144	745	
	600	280F
B224	495	
	256	100G
	□A	

(7)	□B	
0	96	0
4	88	3
8	67	8
7	56	6
8	40	7
	□A	

(8)	□B	
0	100	
8	81	
12	64	
16	49	
12	36	
16	25	
	□A	

(9)	<input type="checkbox"/> B	
6	32	4
0	29	2
5	19	3
7	11	4
4	6	3
5	0	0
	<input type="checkbox"/> A	

11. Lay down a field and find its area from the following notes, the chain lines being all within the field.

Find the length of the tie-line CD, D being on the chain line AB, and find also the perpendicular from C on AB.

(1)	<input type="checkbox"/> A		(2)	<input type="checkbox"/> A	
0	638		1,069	0	
46	465		1,000	30	
32	246		900	20	
0	000		500	30	
	<input type="checkbox"/> C		000	0	
	<input type="checkbox"/> C		<input type="checkbox"/> C		
0	462		<input type="checkbox"/> C		
45	300		1,621	0	
32	150		1,600	20	
0	000		1,200	50	
	<input type="checkbox"/> B		800	30	
	<input type="checkbox"/> B		000	0	
0	725		<input type="checkbox"/> B		
32	565		<input type="checkbox"/> B		
40	400		2,152	0	
25	250	D	2,000	30	
0	000		1,182	D	
	<input type="checkbox"/> A		600	20	
			100	10	
			000	0	
			<input type="checkbox"/> A		

III. Draw a rough plan, and find, in acres, roods and poles the area of a field from the following field-notes in which the length are given in links.

(1)	□ A	
	1,500	0
	1,125	30
0	750	
30	375	
0	000	
	□ C	
	□ C	
0	1,200	
40	900	
	600	0
	300	40
	000	0
turn	□ B	
	to the	right
	□ B	
	900	0
	600	20
	300	10
	000	0
	□ A	

(2)	□ A	
	464	0
	332	50
	□ C	
	□ C	
0	420	
38	330	
	240	0
	124	25
	□ B	
turn	to the	left
	—	
	□ B	
0	572	
26	236	
From	□ A	go North

(3)	450A	0	(4)	<input type="checkbox"/> A		
	350	40		0	5,250	
	200	0		40	4,500	
	0	25		60	3,500	
	<input type="checkbox"/> C			50	2,500	
				0	0,000	
	270C			<input type="checkbox"/> C		
	180					
	100	0		<input type="checkbox"/> C		
	50	40		0	3,460	
turn	000	0	turn	20	3,200	right
	<input type="checkbox"/> B			30	3,000	
	to the	left		20	2,500	
				10	1,500	
	360B	0		0	0000	
	260	20		<input type="checkbox"/> B		
	120	30		to the		
	000	0				
	<input type="checkbox"/> A	go N E.		<input type="checkbox"/> B		
				2,210	0	
From				2,000	20	
				1,500	30	
				1,000	20	
				0000	0	
				<input type="checkbox"/> A		

IV. Show the alterations in the plan and area of the four fields in Ex. III, if we read 'left' for 'right' and *vice versa*

V. Explain fully the various uses of offsets in field measurements.

VI. Lay down the fields and find their areas from the following field notes remembering that you always turn to the *left* after arriving at successive stations and in (1) the perpendicular from B on AC is 160, that from D on AC is 322, and that from E on AD is 146; and in (2) the perpendicular from Q on PR is 184, from S on PR is 296 and from T on PS is 200.

(1)		(2)	
<input type="checkbox"/> D		<input type="checkbox"/> S	
540		480	
<input type="checkbox"/> A		<input type="checkbox"/> P	
<input type="checkbox"/> C		<input type="checkbox"/> R	
600		500	
<input type="checkbox"/> A		<input type="checkbox"/> P	
<input type="checkbox"/> A		<input type="checkbox"/> P	
320	0	350	0
000	0	300	32
<input type="checkbox"/> L		140	30
000		000	0
<input type="checkbox"/> E		<input type="checkbox"/> T	
300	0	<input type="checkbox"/> T	
200	20	282	0
160	0	210	26
100	22	110	0
000	0	000	0
<input type="checkbox"/> D		<input type="checkbox"/> S	
<input type="checkbox"/> D		<input type="checkbox"/> S	
360	0	320	0
340	16	250	0
300	22	200	24
210	10	90	20
160	0	40	0
000	0	000	0
<input type="checkbox"/> C		<input type="checkbox"/> K	
<input type="checkbox"/> C		<input type="checkbox"/> R	
330	0	300	0
320	40	170	40
140	10	100	20 and 24
100	36	60	0
000	0	000	0
<input type="checkbox"/> B		<input type="checkbox"/> Q	
<input type="checkbox"/> B		<input type="checkbox"/> Q	
350	0	320	0
210	40	260	26
120	20	230	0
000	0	200	0
<input type="checkbox"/> A		140	0
		100	0
		000	0
		<input type="checkbox"/> P	

VII (a) Indicate clearly the steps by which the lengths of tie-lines joining definite points in the sides of a triangle can be found geometrically.

(b). In the following field find the area in acres and the length of the tie-line MN.

N	□A	
	3,500	0
	1,340	
	1,000	40
	0000	0
	□C	
	2,980	0
	1,987	40
	0000	0
	□B	
M	□B	
	3,120	0
	2,978	40
	1,500	
	0000	0
	□A	

VIII. Required the plan and area of a field ABCDE from the following notes.

C308	□D	
	1,165	
	912	168E
	415	
	□B	
turn	to the	right
	□B	
	1,195	
	293	233E
	□A	

IX. Make a rough sketch of the field ABC. Calculate its area excluding that of the triangular garden LMN the offsets of the corner LMN from the chain line AC being 13, 79, 113 respectively, and from the chain BC, 15, 86, 124. [The chain lines are all within the field].

(1)	□A	(2)	□A
o	225	o	364
3o	180	75	200
o	000	o	000
	□C		□C
	□C		□C
o	272	o	627
4o	160	90	450
o	000	o	000
	□B		□B
	□B		□B
o	353	o	725
6o	100	50	360
o	000	o	000
	□A		□A

X Lay down the field ABCDEFG and find its area from the following dimensions. Give a geometrical construction for determining the position of the station F. (When the figure is drawn the area can be found mentally).

G128	□D 1,320 737 251 □F	72E
D400	□G 1,445 379 □C	
B124	□C 1,790 1,339 347 □A	76G

SECTION III.

MISCELLANEOUS PROPOSITIONS.

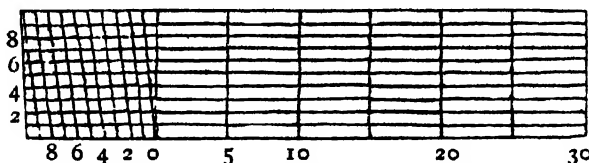
118. **Scales.**—A scale is a conveniently numbered aggregation of equal parts of some distance that has been assumed to represent the unit of the scale

The expression 4 miles to the inch (written 4 miles = 1 inch) means that one inch in the plan represents 4 miles in the ground surveyed and plotted

119 Scales are of three kinds, viz —simply divided scales, diagonal scales and vernier scales

120. **Decimal Diagonal Scale.** Figure 49 represents a decimal diagonal scale.

Fig. 49



The part between the marks 0 and 30 is a simply divided scale divided into six equal parts.

The rectangle to the left of the mark 0 has all its sides divided into ten equal parts, and the points of division are joined by diagonal straight lines longitudinally and by non-diagonal straight lines transversely.

By means of this scale we can measure the relative lengths of lines to two places of decimals. Thus to take off the line represented by 20.6, set one foot of the compass on 20, and extend to the number 6 in the diagonal scale.

To take off the line represented by 10.86, set one foot of the compass on 10 and glide it up to the intersection of the longitudinal line at 10 and the transverse line through 6, then extend transversely to the diagonal line 8. This will give the required distance 10.86.

121 To measure the breadth of a river. Let AD represent the breadth of the river. Fix an object A on the opposite bank. Lay down the line DB in the same straight line with A. Measure off a convenient length DC at right angles to AB (art. 112). Fix the direction CA, and draw the perpendicular CB by means of the chain (art. 112). Then $AD = CD^2 \div BD$ (art. 53), but CD, DB are of known lengths. Hence AD is easily found.

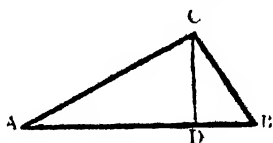


Fig 50

ANSWERS.

EXERCISE XX

- I.—(1) 1 ro. 10 88 perches. (2) 30 68 perches
 (3). 1 ro 21 76 perches. (4). 1 ro 6 86 perches
 (5) 2 ro. 8 32 perches (6) 2 ac 2 ro. 30 perches
 *nearly
 (7). 953 square links (8) 992 sq links. (9) 262½ sq. links

- II.—(1) area = 1 ac. 3 ro. 24 perches nearly; CD = 471 links nearly, perpendicular = 401 links nearly
 (2). CD = 901, perpendicular = 780. area = 7 ac. 1 ro. 22 perches nearly

III.—(1). 5 ac. 1 ro. 9·6 perches. (Some areas cancel).

(2). 99840 sq. links.

(3). 2 ro. 14·88 perches.

(4). 29 ac. 14 perches nearly.

VI.—(1). 2 ac. 0 ro. $5\frac{1}{2}$ perches nearly.

(2). 1 ac. 3 ro. $4\frac{1}{2}$ perches nearly.

VII.— $45\frac{3}{5}$ acres ; 1740 links.

VIII.—4 ac. 0 ro. 26 perches nearly.

IX.—49,911 sq links. (2). 202,225 sq. links.

X.—6 acres.

ALLAHABAD UNIVERSITY PAPERS. 1889.

1. Find the cost of lining a rectangular cistern, 12 feet 9 inches long, 8 feet 3 inches broad, and 6 feet 6 inches deep, with sheet lead weighing 8 lb. per square foot and which costs £1. 8s. per cwt.

2. A tower, which stands on a horizontal plane, subtends a certain angle at a point 160 feet from the foot of the tower. On advancing 100 feet towards it, the tower is found to subtend an angle twice as great as before. What is the height of the tower?

3. The sides of a five-sided figure ABCDE are AB=25 feet, BC=29 feet, CD=39 feet, DE=42 feet, and EA=27 feet. Also AC=36 feet, and CE=45 feet. Find its area.

4. Draw a plan, and find the area of a piece of land, from the following notes :—

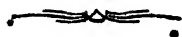
	<input type="checkbox"/> A	
○	672	
72	416	○
	294	12
	226	
○	142	
38	55	
Turn	to the	right
	<input type="checkbox"/> C	
	<input type="checkbox"/> C	
○	640	
60	543	
○	305	
30	220	
Turn	to the	right
	<input type="checkbox"/> B	
	<input type="checkbox"/> B	
	416	○
	364	15
	200	25
	○	○
	<input type="checkbox"/> A	go N. E.

1890.

1. A ladder 24 feet long stands upright against a wall, how far must the bottom of the ladder be pulled out so as to lower the top 3 feet?

2. Find the diameter of the circle round a triangle whose sides are 123, 122, and 49.

ANSWERS.



EXERCISES I. *p.* 9.

1. 1625 sq. yds. 2. 65 sq. ft. 90 sq. in. 3. 9 sq. ft.
4. 13 ac. 1 ro. 23.96 perches nearly. 5. 10 ac. 3 ro. 37.2864
perches. 6. 78 ac. 0 ro. 39.0144 ps. 7. 146.9336 acres. 8. 10
chains. 9. $61\frac{8}{17}$ yds. 10. 368.3 yds. nearly. 11. 41616 sq. in.
13. 54 ft. 14. 72 inches. 15. $30\frac{1}{2}$ sq. ft. 16. $2\frac{1}{2}$ sq. in.
17. 1149 sq. ft. 18. 27 inches. 19. 25 poles and 50 poles.
20. $407\frac{11}{27}$ 21. 4614 sq. yds. 22. 88 yds. 23. 51 acres
2 roods 24 perches 14 sq. yd. 24. 636 acres. 25. 147.

EXERCISES II. *p.* 12.

1. 4 ft. 7' 6". 2. 7 ft. 2' 9". 3. 5 sq. ft. 10' 11" 6" 4. 9 sq. ft.
3' 0" 3". 5. 111 sq. ft. 0'. 6. 93 sq. ft. 11' 0" 4". 7. 32 sq. ft. 9'
5" 6". 8. 7 ft. 8 $\frac{1}{2}$ inches. 9. 5 sq. ft. 53 $\frac{1}{2}$ sq. in. 10. 300 sq. ft.
69 $\frac{7}{12}$ sq. in.

EXERCISES III. *p.* 13.

1. 7 sq. ft. 10' 6". 2. 15 sq. ft. 9' 11". 3. 48 sq. ft. 4' 11"
11" 4". 4. 8 sq. yds. 1 sq. ft. 2' 2". 5. 17 sq. yds. 5' 9"
6" 8".

EXERCISES IV. p. 17.

1. Rs. 6900 13as 4 pie; 1271961 $\frac{3}{5}$; 11041. 2. $277\frac{1}{27}$ sq. yds.
 3. $387\frac{5}{8}$ yds. 4. 13900 sq. ft. 5. 25 chains. 6. 37600.
 7. $7\frac{25}{27}$. 8. Rs. 36-1-9 $\frac{3}{5}$ pie. 9. Rs. 19-11 as. 4 pie, and Rs.
 21-14 as. 10. $42\frac{1}{2}$ yds; Rs. 112-14 as. 3 pie. 11. 15 inches.
 12. 9 ac. 2 ro. 28.92 perches nearly. 13. $149\frac{1}{2}$ sq. ft. 14. 360sq. ft.
 15. 9 yds. 16. Breadth = 20 ft.
 height = 16 ft.
 17. 50 minutes.
 18. 943,942,400 acres.
 28,555,825,760 bighas.
 20. Length = 25 ft.
 breadth = 15 ft.
 height = 12 ft.

EXERCISES V. p. 22.

1. 205. 2 353. 3. 1157. 4. 72. 5. 609. 6. 464.
 1205. 8. 80. 9. 22440. 10. 33461.

EXERCISES VI. p. 28.

1. (1) 41 inches. (2) 18 ft. 5 inches. (3) 58 ft. 1 in.
 (4) 99 ft. 1 inch. 2. (1) 2 ft 9 inches. (2) 2 ft. 7 inches.
 (3) 9 yds. 0 ft. 9 inches. (4) 15 yds. 1 ft. 7 inches. 3. $m^2 + n^2$.
 4. one acre. 5. 918750 sq. ft. 6. $120\sqrt{3}$ yds. 7. 9.348
 acres. 8. 121 miles. 9. 20 chains. 10. 90, 400, 410.
 11. 60. 12. 200. 13. 353.55 yds. 14. 34 ft. 15. $9\frac{1}{2}$ yds.

16. (1) 7.071 ft. (2) 11.401 ft. (3) 32 ft. 9.39 inches nearly.
 (4) 607.22 inches. 17. 65 ft. 18. 12 ft. 19. 25 ft. 20. 42.5 cubits
 21. 2 ft. 22. 7 ft. 23. 44 ft. 24. 31.57 yds. and 8 yds. 25. 39.6 ft.
 and 84.7 ft. 26. 267 miles. 27. 313 yds. 28. 1625 ft. 29. 2016
 sq. ft. 30. $\sqrt{51}$ cubits. 31. 71 miles. 1 furlong. 32. 6 ft. 1 inch.
 33. 13.69 sq. ft. 34. base = 120 ft. alt = 119 ft. 35. $3\frac{3}{4}$ cubits.
 36. 12 cubits. 37. 50 cubits. 39. sides are 6.5, 6.5, 6.6 and 6.5,
 6.5, 11.2

area = 18.48.

EXERCISES VII. p. 37.

I. (1) 24 sq. ft. (2) 1116 sq. yds. (3) 30600 sq. lks.
 (4) 1760 sq. yds. (5) 1160250 sq. ft. (6) 370675800 sq. lks.
 II. (1) 9.797. (2) 142.912. (3) 971.658. (4) 18869.112

Example 2. p. 40.

(1). Acute-angled. (2). Obtuse-angled. (3). Obtuse-angled
 (4). Right-angled. (5). Acute-angled.

EXERCISES VIII. p. 41.

1. 15625 $\sqrt{3}$ sq. ft. or 27063.29375.. sq. ft. nearly. 2. $327\frac{1}{5}$
 sq. ft. 3. 7500 $\sqrt{15}$ sq. chains or 2900 $\frac{1}{4}$ acres nearly. 4. 7392
 sq. yds. 5. 1848 sq. ft. 6. 1219 sq. ft. 7. 102 acres.
 8. 16 $\sqrt{3}$ sq. yds. 9. Rs. 23100. 10. 600 yds. 11. 120120
 sq. yds. 12. 5773 $\frac{1}{2}$ sq. ft. nearly. 14. 762.372 sq. ft.
 15. 533,533,616. 16. $21\frac{9}{13}$ ft. 17. $4\frac{13}{15}$ ft. 18. 709 ft. nearly.
 19. £ 4. 20. 240 yds. 21. $28\frac{4}{11}$ sq. ps. 22. 583. 23. $73\frac{2}{11}$ ac.
 24. 1 ch. 743 $\frac{97}{121}$ lks. 25. 12 ft. 26. 93947 $\frac{2}{9}$ sq. yds. nearly.

27. Rs. 407-5 as. 28. 29 yds. 29. 15,396 yds. nearly.
 30. 340 yds. 31. $118\frac{5}{22}$ acres. 32. 13821.1428 sq. ft. 33. 171120
 sq. yds. 34. 78, 89, 89. 35. 12450.8 sq. ft. nearly. 36. 114
 yds. 37. (1) acute-angled (2) obtuse-angled (3) obtuse-
 angled (4) acute-angled (5) right-angled (6) acute-angled.
 38. 21.07, 40.77, 57.5. 39. 99, 15. 40. 396, $173\frac{3}{5}$, $212\frac{71}{101}$. 41. 110
 sq. yds. 42. 2574, $5238\frac{1}{2}$, 968, $6844\frac{1}{2}$.

EXERCISES IX. *p.* 46.

1. 13125 sq. yds. 2. 19,3649 ft. 3. 440 ft. 4. 16100 sq. yds.
 5. 7140 sq. ft. 6. 435 sq. ft. 7. 375 sq. ft. 8. 234 acres
 1 rood 20 perches. 9. 2520 sq. ft. 56 ft. 10. 32.5 ft. 1032.24
 sq. ft. 11. 384, 32. 12. 352 yds. 13. $9\frac{1}{2}$ yds. 14. 77 yds.
 15. 1248 yds. 16. $62\frac{1}{2}$ ft. $93\frac{3}{4}$ ft.

EXERCISES X. *p.* 50.

1. 681200 sq. lks. or 6 ac. 3 ro. 9.92 perches. 2. Rs. 341666
 - 10 as. - 8 p. 3. $41\frac{3}{8}$. 4. 198 sq. yds. 5. $78\frac{1}{8}$. 6. 27 ft.
 23 ft. 7. $114\frac{2}{27}$ yds. 8. 216. 9. 80 lks. 10. 7 ft. 11. 48013
 sq. ft. 12. 49470 sq. yds. 13. 540 sq. yds. 14. 40800 sq. ft.
 15. 45875 sq. yds. 16. $8\frac{1}{4}$, $8\frac{1}{4}$, $9\frac{1}{4}$, $9\frac{1}{4}$ sq. metres.

EXERCISES XI. *p.* 53.

1. 35520000 sq. yds. 2. 1687.5 sq. yds. 3. 128000 sq. yds.
 and 42240 sq. yds. 4. 660 yds. 5. 10125 chains. 6. 720
 sq. yds. 7. $1154\frac{1}{4}$ sq. yds. 8. 12600 sq. yds. 9. 6250 sq. ft.

10. 21622. 11. 134055 sq. ft. nearly 12. 3.25 sq. chains.
 13. 954. 14. 35046 sq. yds, 15. $3984\frac{1}{2}$ sq. ft. $5304\frac{1}{2}$ sq. ft.

EXERCISES XII. *p.* 56.

1. 592.8 acres 2. 26105 sq. chains. 3. 4890 sq. ft.; 37 ft.
 4. 18840 sq. yds. 5. 4108 sq. ft. 6. $4110\frac{1}{2}$ sq. ft. 7. 13848
 sq. yds. 8. 126195.5 sq. ft. 9. 584415 sq. lks. 10. 12660000
 11. 1044.5 sq. ft. 12. 587 sq. ft.

EXERCISES XIII. *p.* 60.

1. (1) 60° (2) 90° (3) 108° (4) 120° (5) $128\frac{4}{7}^\circ$ (6) 135° (7)
 140° (8) 144° (9) $147\frac{3}{11}^\circ$ (10) 150° (11) 156° (12) 162° . 2. (1)
 $27'$ (2) $6'45''$ (3) $1^\circ 48'$ (4) $30^\circ 30'36''$ (5) $32.4''$. 3. (1) .472 (2)
.0445 (3) .492. 6. 16. 7. 30° 8. $150^\circ 21' 43''$ 9. $176^\circ 24'$
10. $46^\circ 22' 2''.5$ 11. $72^\circ 24' 6''$ 13. $37^\circ 22'$; $52^\circ 38'$.
14. 60° .

EXERCISES XIV. *p.* 68.

1. $16406\frac{1}{2}$ sq. ft. 2. $15250\sqrt{3}$ sq. ft. 3. $174\sqrt{2}$ sq. chains.
 4. 30 acres. 5. 1443.3756 sq. ft. 6. $\frac{1}{2}\sqrt{1753}$; $16(\sqrt{2})$
 $1175\sqrt{2}$ sq. ft. 8. $4906\frac{1}{2}$ sq. ft. 11. $2500\sqrt{3}$.
 12. $128\sqrt{3}$ sq. yds. 13. 1215.625 sq. chains. 14. 50 ft.
 15. 2232 sq. ft. 16. 41.231056 sq. ft. 17. 13680 sq. ft., 184.17
 ft. nearly. 18. 1120 sq. yds. 19. $14\frac{2}{5}$ ft. 20. 132.8 ft. nearly.

MISCELLANEOUS EXAMPLES p. 70.

1. 95780 sq. yds. 2. 80 miles. 3. 650 sq. ft. 4. 200 ft.
 5. 120 sq. yds. 6. $140\frac{5}{8}$. 7. 1760. 8. 3000 sq. ft. 9.
 Rs. 12-5 as, 3 pie. 10. 461 ft. 11. 455,545. 12. 79260.74 acres.
 13. 20 $\sqrt{10}$ yds. long, 5 $\sqrt{10}$ yds. broad, 10 $\sqrt{10}$ yds. high; Rs.
 540. 14. 866250 sq. ft. 15. Rs. 36-4 as. $9\frac{3}{5}$ p. 16. 514976.
 17. 66192. 18. $67\frac{1}{2}$ ft. 19. $60\sqrt{6\sqrt{3}}$ ft. 20. 67644.
 21. 296 ft. 22. $26\sqrt{3}$. 23. 2500 sq. ft. 24. 96800
 25. .0453 sq. miles nearly. 26. $6\frac{2}{5}$ ac. $3\frac{3}{5}$ ac. 27. 1350 sq. ft. $37\frac{1}{2}$ ft.
 36 ft. 28. $315\frac{7}{15}$ yds. 29. 2400, 2600, 3200, 1800. 30. 9057.6.
 31. $13\frac{4}{5}$ minutes. 32. 1054 ft. 625 ft. $566\frac{394}{625}$ ft. 33. 44 sq. ft.
 7'-11"-8" 10"" 34. 107.1 sq. yds. 35. 16 £ 3 s. $6\frac{1}{2}$ d.
 36. $3\frac{1591}{2178}$ ac. 269 yds. 1 ft. 4 in. 37. 25 ft., 10 ft. 20 ft. 38. 276.125
 sq. ft. 39. 31,21,13. 40. $26\frac{1}{24}$ sq. ft. 41. 12,000 yds. 42. $193\frac{101}{363}$.
 43. 12637,12012. 44. Rs. 4 10 as. 8 pie. 45. 14.62 acres nearly.
 46. $36\frac{9}{24}$ sq. yds. 47. 81.5085 sq. ft. 48. 33 rounds. 49. £4. 50 $25\frac{5}{9}$
 yds. Rs. 28 12 as. 51. 3091 yds. 52. $200(\sqrt{7}+1)$ and
 $200(\sqrt{7}-1)$ ft. 53. 1430. 54. $550\sqrt{5}$ sq. ft. 55. 2 £
 15 s. 56. 63300 sq. lks. nearly. 57. 384 sq. ft. 58. $20\sqrt{3}$,
 $300\sqrt{3}$ sq. ft. 59. 8.13 nearly. 60. £1218-19 s. $3\frac{1}{9}$ d.
 61. side = $3\frac{1}{2}$. 62. $18\sqrt{15}$, $162\sqrt{15}$. 63. 18.439 yds. 64. 18.
 65. 21.1 nearly. 66. 5000. 67. 26.883 acres nearly. 68. 2240. 69.

- 1 ro - $26\frac{2}{3}$ per. **71.** $2541\sqrt{15}$ sq. ft. **72.** 200 sq. ft. **73.** $40\sqrt[4]{10^8}$
 or 128.9 nearly. **74.** 12 ac. 2 ro. 39.4 per. **75.** 33 £.
76. £98454 - 10s. - $10\frac{10}{11}d.$ **77.** 1 ft., 2 ft. **78.** 64 ft. nearly. **79.**
 £10 11 s. 9d. **80.** $270, 150\sqrt{10}$ sq. ft. **81.** 25 ft. **82.** Same
 as question No. 50. **83.** 17.644 ft. nearly. **84.** 96 ft.
85. 96. **86.** 528 yds. **87.** 20.8924 ft. **88.** 315 ac., $78\frac{1}{2}$ ac.,
 $236\frac{1}{2}$ ac. **89.** 24, 36, 16. **90.** 210, 294 sq. ft. **91.** $75\frac{23}{27}$ yds.
92. 20 ft. **93.** 31828 sq. ft. **94.** 1074 ac. 0 ro. 24 per.
95. 353.268 sq. ft. **96.** 10 ft. **97.** $455\sqrt{3}$. **98.** 2343.75 sq. yds.
99. $1250\sqrt{3}$ sq. ft. **100.** 30 yds. **101.** 1418.5 sq. ft.
102. 80 ft. $54\frac{1}{2}$ ft. (nearly). **103.** $1087\frac{1}{2}$ sq. ft. **104.** 1732 ft.
 $10\sqrt{21}$ ft. $10\sqrt{61}$ ft. **105.** 42 ft. 7 in. (nearly). **106.** 25.22.
107. 19.56312, 385.43688. **108.** 806.22577 lks. **110.** $3\frac{3}{7}$ and $4\frac{4}{7}$ -
111. 1.732 miles. $1\frac{1}{2}$ miles. **112.** $30\sqrt{3}$. **113.** 3.35 acres.,
114. 1872, 74.88, 48, 46.8. **115.** 1.3660254.....miles. **116.**
 227.35 sq. ft. **117.** 4675 sq. yds., 65 yds. **118.** 15 ft.
 54 sq. ft. **119.** Icosagon. **120.** 3 ac. 2 ro. $14\frac{73}{121}$ po.
121. 200 ft., $\frac{19775}{313632}$. **122.** 156.2 ft. nearly. **123.** 23.8. **124.**
 12.29, 10.295.

PART II.

EXERCISES XV. p. 92.

- 2.** .63503. **3.** 29 ft. **4.** 187 inches. **5.** 33.8 ft. **6.** 54.27 ft.
 nearly. **7.** 9 ft. **8.** 18.8885 miles. **9.** 50 ft. and 25 ft.
10. 10 ft. **12.** 12.8452326 miles. **13.** 50 ft. **14.** 36 ft.
15. 266664 ft. **17.** 5.4272034 ft. and 3.3166248 ft. **18.** 54.1875.

19. 80 ft., 46.188 ft. and 92.376 ft. 20. 61 ft. & 388.27 ft.
 21. 132.287565 ft.; 173.20508 ft.; 193.649165 ft.; 200 ft.;
 193.649165 ft.; 173.20508 ft. and 132.287565 ft.

EXERCISES XVI p. 97.

1. $R = 8\frac{1}{8}$ ft.; $46\frac{1}{24}$ yds. $288\frac{9}{240}$ lks. $602\frac{1}{2}$ ft.; $196\frac{1}{2}$ ft. 235250
 lks. $\frac{35}{24}\sqrt{6}$; $\frac{6125}{20424}\sqrt{5106}$; $\frac{35211}{216\sqrt{1221}}$; and $\frac{1554525}{216\sqrt{1221}}$. $r =$
 $1\frac{3}{8}$ ft.; 12 yds.; $56\frac{1}{4}$ lks. $49\frac{1}{8}$ ft.; 1400 ft.; 19300 lks.; $\frac{1}{2}\sqrt{6}$; $2\sqrt{\frac{138}{37}}$.
 and $\sqrt{\frac{47}{62}}$; and $\frac{4}{3}\sqrt{1221}$. $r_a = 2$ ft.; $20\frac{2}{3}$ yds.; $226\frac{2}{3}$ lks.; 352 ft.
 26520 ft.; 1980 lks.; $\sqrt{6}$; $2\sqrt{\frac{222}{23}}$; $18\sqrt{\frac{31}{94}}$; and $\frac{45}{22}\sqrt{1221}$. $r_b = 8$ ft.;
 36 yds.; $81\frac{3}{5}$ lks.; 90 ft.; 2380 ft.; 96030 lks.; $\frac{4}{3}\sqrt{6}$; $2\sqrt{\frac{851}{6}}$; $\frac{2\sqrt{2914}}{3}$;
 and $540\sqrt{\frac{11}{111}}$. and $r_c = 24$ ft.; $139\frac{1}{2}$ yds.; 900 lks.; 160 ft.; 3900 ft.;
 374420 lks.; $4\sqrt{6}$; $\sqrt{5106}$; $6\sqrt{2914}$; and $18\sqrt{1221}$. 2. 125 ft.
 3. 24 sq. ft.; 10 ft.; and 2 ft. 4. 57.735 ft. 5. 293. 6. 4.
 7. 4 and 8.944272.

EXERCISES XVII. p. 106.

1. 1.732 yds. 2. 8.48526 ft. 3. Side = circumradius
 = 12.3 nearly. In-radius = 10.7 nearly. 6. 2338.2 sq. ft.
 7. 64.95 sq. inches. 8. 4.680 ft. nearly. 9. 506.25 sq. inches.
 10. 40.5 sq. ft. 11. 93.528. 12. 180 ft.

EXERCISES XVIII. p. 110.

1. (1) 62.8318 ft. (2) 34.55749 ft. (3) 6.28318 miles.

2. 50400 Revolutions. 3. 91 yds. 4. 161407.71 miles nearly. 5. 1047.19755. ... 6. 11.25 miles. 7. 240 trees. 8. 330 yds. 9. 35 yds. 10. 3.18309 feet. 11. 105 yds. 12. 70.71 ft. 13. 314.159. 14. 2830.57259. 15. 999.29 inches. 16. 2.1008394 ft. nearly. 17. 29.370629 in. 18. 15.9654 in. 19. 22.50762939 ft. 20. 3125 miles. 21. 28 ft. nearly. 22. 385 ft. 23. 84 : 5. 24. 218.75 yds. nearly. 25. 21560 ft.

EXERCISES XIX p. 120.

1. 65 ac. - 3 ro. - 2.640192 per. 2. Rs. 22192 as. 8 nearly. 4. 7.98 ft. nearly. 5. 1; $\sqrt{2}$ or 1.4142136; $\sqrt{3}$ or 1.7320508; and 2 inches nearly. 6. 10.27431 sq. ft. 7. 28:3. 8. 28.54. 9. 117.57 sq. chains. 10. 6679.951 sq. ft. 12. 21.46018366 sq. inches. 13. .6 sq. inches nearly. 14. (1) 562500 sq. ft., (2) 649519.05 sq. ft. nearly (3) 716195.25 sq. ft. 15. $\frac{m^2(4-\pi)}{4}$. 16. 60. 17. 9 sq. ft. - 36 sq. inches nearly. 18. 27.4657375414768 nearly, 1938.36103 ft. 19. 31415.926536 sq. ft. 20. 255.15 sq. inches. 21. 31.8309 sq. ft. 22. 305362.548 sq. ft. 23. £ 2 - 5 s. nearly. 24. 9173.4428 yds. 25. 86140 sq inches. 26. 2114.159 sq. ft. and when the chain increases, the area is increased by 2092.477 sq. feet.

MISCELLANEOUS EXAMPLES, p. 126.

1. 51.05. yds. 2. $5\sqrt{10}$ ft. 3. 185 ft. 4. 5 miles approxi-
mately. 5. $25\frac{11}{13}$, 24, $22\frac{2}{5}$ yds. 6. 12 ft., 8 ft. 7. 27 sq. ft.
and 22.83. sq. ft. 8. Altitudes are 3.641, 5.259, 7.886, Medians

- are 4.031, 9.069, 10.7703, $R = 7.4127$. 10. 120. 12. 609.9648 sq. ft. 13. $1\frac{1}{2}$. 14. 204.12415 ft. 15. 26.4. 16. $4:3 \sqrt{3}:6$. 17. $4 \sqrt{3}$ yds. 18. 49, 168 ft. 19. $112\frac{1}{2}$ ft. 20. $3\frac{6}{11}$ ft. $19\frac{1}{2}$. 21. 180, $360 \sqrt{5}$. 22. $40 \sqrt{3}$, 40 ft. 23. 13. 24. $59\frac{781}{901}$ ft. 25. 14, 936. 26. 75.39816 yds. Shortest and longest distances from the vertex opposite to the side 97 = 7.4, 41.4 yds respectively. 27. 12 ac. - 0 ro. - 11.37048 perches. 28. $75\frac{3}{7}$ ft. 29. $150 \sqrt{6}$ yds. 30. 2640 ft. 31. 148.0145 and 24.3215. 32. 86.6025 miles. 33. 20 sq. in. 4.43 inches and 11.29 in. 34. 1.5811388 3.391165 and 2.183882. 35. (1) $mn (m^2 - n^2)$ (2) $pqr \times (qr + ps)(pr - qs)$ (3) $2a(a+2)(2a-1)$. 36. 200, $100 \sqrt{3}$. 37. $\Delta = 19206330$; $r = 1815$ $R = 4383.5$. 38. 30° , 90° , 60° . 39. 780. 40. $(10 + 3 \sqrt{5})$, $(10 - 3 \sqrt{5})$. 41. 74.88, 48, 46.8. 42. 161.5. 43. 15.1306807. 44. 100 ft. nearly. 45. $\frac{51}{4} \sqrt{91}$. 46. 14400:14121. 47. 3120. 48. 1140. 49. 64, and 44. 50. 2.64. 51. 6.0207973, 7.7.7620873. 52. 3720. 53. 112.2. 54. 2:1, 3:1 55. $4 \sqrt{30}$. 56. 84. 57. 124. 58. 6.2 ft. nearly. 78.22946 miles. 60. 2048718.75. 61. 21.99, 14.49, 8.77 nearly. 62. $r = .4375$ $R = 1.200625$. 63. 49.725. 64. 162.5, 87.36. 65. 4.1° and .9. 68. 102.5. 69. 8000 miles approximately. 70. $266\frac{2}{3}$ ft. 71. 78.41. 72. $210\frac{13}{42}$. 73. $\frac{32}{15} \sqrt{15}$, $\frac{16}{5} \sqrt{15}$, $\frac{8}{5} \sqrt{15}$. 74. $R = 40.625$. 75. 16.25. 76. 8, and 6. 77. 189 sq. ft. 78. 1:4. 79. 2.5. 80. 6; 2.5. 81. 20 metres. 82. 2.8. 83. $18\frac{6}{13}$. 84. 420, 7, 325. 85. 1:4. 86. $4 \sqrt{46}$. 87. 1896. 88. 46.2711. 89. 135° . 90. 6.

- 92.** 15, 20, 25. **93.** 9, $20\frac{5}{6}$. **94.** 279075. **95.** 15. **96.** 24.
97. see art. 76. **98.** 29.75. **99.** 14.625. **100.** $221\frac{13}{16}$ **101.** 221.
102. 78. **103.** 22. **104.** 150° . **105.** 112. **106.** $7\frac{7}{8}, 4\frac{7}{8}, 3\frac{1}{8}$. **107.** 9.4339811, 17.888544, 84.3800924. **108.** $300\sqrt{3}$. **109.** 3.2, 1.8. **110.** $221, 78\frac{6}{13}$. **111.** 455488.561783684. **112.** 21.87692
 52.5045, 9.1155. **113.** $42\frac{54}{203}$ perches. **116.** 310.582 ft. **117.** 19855.6974 sq. in. **118.** 25 ft. **119.** 47.12385 ft. **120.** 2100 sq. ft. **121.** 134.164079 miles. **122.** $10\sqrt{3}$, 5. **123.** $289\frac{12}{29}$, 42000. **124.** 714. 15926536...sq. ft. **125.** 4 ft. **126.** Rs. 241 - 6 as. - $8\frac{16}{25}$ p. **127.** $512(\sqrt{2}-1)$ or 1212.0773632. **128.** $\sqrt[4]{27}:2$
129. 4 inches. **130.** $69\frac{3}{7}$ sq. in. **131.** $36^\circ, 72^\circ, 72^\circ$. **132.** 80 yds. **133.** 1875 sq. ft. **134.** $62\frac{6}{7}$ ft. $94\frac{2}{7}$ ft. **135.** 26 ft. **136.** 81, 1640. **137.** 1, and 7. **138.** $9:4\sqrt{3}$. **139.** 42.5 in.
140. $885\frac{1}{3}$ sq. ft. **141.** $66\frac{4}{13}$ ft. **142.** 30, 40. **143.** $3750\sqrt{3}$, 5000 $\sqrt{3}$. **144.** 3 sq. ft. **145.** 75, 180, 195. **146.** ~~$\sqrt{3}:2$~~
147. 314.15926536 sq. ft. **148.** $4(3\sqrt{2}-2\sqrt{3})$ sq. ft. **149.** Rs. 311 nearly. **150.** 750 sq. yds. **151.** 2, 4, $6\sqrt{3}$.
152. 8284.272. **153.** 15 ft. nearly. **154.** 617.39256 sq. ft. **155.** $6075\frac{\sqrt{3}}{4}$, $6075\sqrt{3}$. **157.** $11\frac{1}{5}, 12, 12\frac{12}{13}$; perpendiculars.
 10.035, 12.16, 12.971, medians. **158.** 79577 sq. yds. nearly. **159.** 1875 sq. ft. **160.** 80.75. **164.** 121 and 40 ft. **165.** 425.
166. .725, 1.075. **167.** bisector 8.1, sides = 8.86, 10.06. **168.** $\frac{1}{85}$

- 169.** .164128. **170.** 12.409, 20.95, 16.91. **171.** 936, 48.
172. 9,40,41. **173.** Rs. 53 - 4 as. nearly, Rs. 600. **174.** 1113, 1625.
175. 191 ac. 1 ro. **176.** 31.25, 25.787 and 19.18. **177.** 8.062,
 7.615, 4.121, 3.162. **178.** 400 ($\sqrt{2} - 1$) or 1.6568544. **179.**
 68.866279 acres nearly. **180.** $3\frac{1}{13}$ ft. **181.** Rs. 333. **182.** area =
 32.832 sides = 10.94, 9.12, 7.296 nearly. **185.** 7.071 ft. **187.**
 4.7 inches. **188.** 12, 40. **189.** 205. **190.** A circle whose area is
 24,649,917,586,034,029, 603,2832..... sq. yd.

ALLAHABAD UNIVERSITY PAPERS.

1889.

1. £36 16s. 4½d. 2. 80 ft. 3. 1602 sq. ft. 4. 152873

1890.

1. 11.62 ft. nearly. 2. 125.05.



Allahabad:

PRINTED AT THE "INDIAN PRESS."

